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TIME-OPTIMAL CONTROL OF A THIRD ORDER REGULATOR

by

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Time-Optimal Control Of
A Third Order Regulator

by

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of the requirements for the degree of

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ABSTRACT

The time-optimal control law as a function of the states for second and third-order linear regulators with real eigenvalues was derived. Notions of a switching curve for the second-order system and switching surface for the third-order system was introduced. A set of states was found which divided the state space into two distinct regions, in one of which the time-optimal control was +1 and in the other of which the time-optimal control was -1.

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I. INTRODUCTION

The study of specific time-optimal systems has been continuing in the development of modern control theory. To solve an optimization problem, we must first define a goal or a cost function (performance measure) for the process we are trying to optimize. With a knowledge of the cost function, and the system states and parameters, we can determine the control which minimizes (or maximizes) the cost function. For some systems a criterion of minimum response time may not be the most suitable measure of system performance.

In this thesis, a minimum-time control for a linear, time invariant third-order regulator with three distinct, nonpositive, real eigenvalues is developed. The time-optimal control, as a function of the states, must transfer the system from any arbitrary initial condition to a target set (origin of the state space) as quickly as possible.

II. MINIMUM-TIME CONTROL

The goal in a minimum-time problem is to transfer the state of the system to a target set as quickly as possible. We assume that the target set is the origin of the state space. For this reason we call this problem the *linear time-optimal regulator problem*. [Ref. 1]

A. PROBLEM STATEMENT

Consider the dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{2.1}$$

where the n-vector $\boldsymbol{x}(t)$ is the state, the system matrix $\boldsymbol{A}$ and the gain matrix $\boldsymbol{B}$ are n x n and n x m constant matrices, respectively. The m-vector $\boldsymbol{u}(t)$ is the control.

We assume that the system (2.1) is completely controllable and that the components of $\boldsymbol{u}(t)$ are bounded in magnitude by the relation

$$|u_j(t)| \le 1 \qquad j = 1,2,\ldots,m \tag{2.2}$$

Given that at the initial time $t_0 = 0$, the initial state of the system is

$$\boldsymbol{x}(0) = \xi, \tag{2.3}$$

we are asked to find the control $\boldsymbol{u}^*(t)$ that transfers the system from ξ to the origin $\boldsymbol{0}$ in minimum time. [Ref. 1]

If the output $\boldsymbol{y}(t)$ of the system (2.1) is related to the state $\boldsymbol{x}(t)$ and the control $\boldsymbol{u}(t)$ by the relation

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \tag{2.4}$$

then a control that drives the states to the origin can be extended in such a way as to drive the output to zero and hold it at zero thereafter [Ref. 1]. If t^* denotes the minimum time required to force the states to the origin, then at $t = t^*$, we have

$$\begin{aligned} x(t) &= 0 \\ y(t) &= 0 \end{aligned} \tag{2.5}$$

where the control is set

$$u(t) = 0 \qquad\qquad \forall\ t > t^* \tag{2.6}$$

B. NECESSARY CONDITIONS FOR THE TIME-OPTIMAL CONTROL

1. Performance Measure

In order to evaluate the performance of a system quantitatively, we must select a performance measure. An optimal control is defined as one that minimizes (or maximizes) the performance measure. Mathematically, the performance measure to be minimized for the minimum-time problems is defined as

$$J = \int_{t_0}^{t_f} dt = t_f - t_0 \tag{2.7}$$

2. Pontryagin's Minimum Principle

The Hamiltonian function for the problem is

$$H[\,x(t),p(t),u(t),t\,] = 1 + p^T(t)Ax(t) + p^T(t)Bu(t) \tag{2.8}$$

where $p(t)$ is the costate vector. Let us assume that a time-optimal control exists and transfers the initial state (2.3) to the origin 0 in minimum time t^*. If $x^*(t)$ denotes the optimal state trajectory of the system (2.1) corresponding to $u^*(t)$, originating at ξ at $t_0 = 0$ and hitting the origin in minimum time, then $x^*(t)$ and $p^*(t)$ need to satisfy the canonical equations given by

$$\dot{x}^*(t) = \frac{\partial H[x^*(t),p^*(t),u^*(t),t]}{\partial p^*(t)} \tag{2.9}$$

or equivalently

$$\dot{x}^*(t) = Ax^*(t) + Bu^*(t) \tag{2.10}$$

and

$$\dot{p}^*(t) = \frac{-\partial H[x^*(t),p^*(t),u^*(t),t]}{\partial x^*(t)} \tag{2.11}$$

or equivalently

$$\dot{p}^*(t) = -A^T p^*(t) \tag{2.12}$$

with the boundary conditions

$$\begin{aligned} x^*(0) &= \xi \\ x^*(t^*) &= 0 \end{aligned} \tag{2.13}$$

The necessary condition for all admissible controls $u(t)\ \forall\ t \in [0,t^*]$ is

$$1 + p^{*T}(t)Ax^*(t) + p^{*T}(t)Bu^*(t) \le 1 + p^{*T}(t)Ax^*(t) + p^{*T}(t)Bu(t) \tag{2.14}$$

In other words a necessary condition for $u^*(t)$ to minimize the performance measure J is

$$H[x^*(t),p^*(t),u^*(t),t] \le H[x^*(t),p^*(t),u(t),t] \qquad \forall\ t \in [0,t^*] \tag{2.15}$$

Equation (2.14) yields the relation

$$u^*(t) = -sign\{B^T p^*(t)\} \tag{2.16}$$

Equation (2.15), which indicates that an optimal control must minimize the Hamiltonian, is called *Pontryagin's minimum principle* [Ref. 2]. Let us now state some important theorems concerning the time-optimal control.

a. Controllability

Controllability is very important, because we consider problems in which the goal is to transfer a system from an arbitrary initial state to the origin while minimizing the performance measure. Thus controllability of the system is a necessary condition for the existence of a solution.

A linear, time-invariant system is controllable if and only if the n x mn matrix

$$\boldsymbol{Q} = [\boldsymbol{B}|\boldsymbol{AB}|\boldsymbol{A}^2\boldsymbol{B}|....|\boldsymbol{A}^{n-1}\boldsymbol{B}] \tag{2.17}$$

has rank n (order of the system). If there is only one control input (m = 1), a necessary and sufficient condition n x n matrix $\boldsymbol{Q}$ to be nonsingular.

b. Observability

A linear, time-invariant system is observable if and only if the n x qn matrix

$$\boldsymbol{R} = [\boldsymbol{C}^T|\boldsymbol{A}^T\boldsymbol{C}^T|....|(\boldsymbol{A}^T)^{n-1}\boldsymbol{C}^T] \tag{2.18}$$

has rank n. If there is only one output (q = 1), a necessary and sufficient condition for observability is that $\boldsymbol{R}$ to be nonsingular.

c. Existence

If all of the eigenvalues of the system matrix $\boldsymbol{A}$ have nonpositive real parts, then an optimal control exists and it is *bang bang*.

d. *Uniqueness*

If an extremal control exists, then it is unique. A control which satisfies the necessary conditions in equations (2.9) through (2.15) is called an extremal control.

e. *Number of Switchings*

If the eigenvalues of the system matrix $\boldsymbol{A}$ are all real and a unique time-optimal control exists, then the control can switch at most $n-1$ times.

III . TIME-OPTIMAL CONTROL OF A SECOND-ORDER PLANT WITH TWO TIME CONSTANTS

A. TIME-OPTIMAL SYSTEMS

The problems we consider in this and the next chapter will involve a single control variable $u(t)$. The systems we examine are time-invariant, and the control is to be a function of the states. Time-optimal control will be a piecewise constant function of time over the sets or regions of the state space. These sets are separated by curves in two-dimensional space, and by surfaces in three-dimensional space. The separating sets are called *switching curves,* and *switching surfaces.* [Ref. 1]

The procedure that will be used in finding the optimal control for both second and third order regulator problems can be outlined as follows :

- Define the problem precisely
- Form the Hamiltonian function
- Find the H-minimal control
- Find the equations of the costate variables
- Determine the control sequences that are candidates for the optimal control
- Determine the switching curves and switching surfaces that divide the state space into various regions
- Find the control sequences that satisfy the boundary conditions
- Simulate the *linear time-optimal regulator* with initial conditions emanating from each possible region of the state space.

1. Problem Definition

We consider the system described by the second-order differential equation

$$\frac{d^2y(t)}{dt^2} + (\alpha + \beta)\frac{dy(t)}{dt} + \alpha\beta y(t) = u(t) \tag{3.1}$$

where $y(t)$ is the output, $u(t)$ is the control which is restricted in magnitude by the relation

$$|u(t)| \le 1 \tag{3.2}$$

and α , β are real, distinct, nonzero eigenvalues. The transfer function of the system is

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{(s+\alpha)(s+\beta)} \tag{3.3}$$

with real poles at $s = -\alpha$ and $s = -\beta$. Using Eqn. (3.3) the state space equations can be written in matrix form as

$$\dot{y}(t) = \begin{bmatrix} 0 & 1 \\ -\alpha\beta & -(\alpha+\beta) \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{3.4}$$

Note that Eqn. (3.4) is of the form

$$\dot{y}(t) = Ay(t) + Bu(t) \tag{3.5}$$

First we need to check the controllability and the observability of the system.

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & -(\alpha+\beta) \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.6}$$

Since there is only one control input and both matrices are nonsingular, the system is controllable and observable. Since the eigenvalues of $\boldsymbol{A}$, $-\alpha$ and $-\beta$ are nonpositive real numbers, an optimal control exists, and it is unique.

Now, we define a matrix $\boldsymbol{P}$ whose columns are the eigenvectors of $\boldsymbol{A}$, and a new dependent variable $\boldsymbol{z}(t)$ by

$$\boldsymbol{z}(t) = \boldsymbol{P}^{-1}\boldsymbol{y}(t) \tag{3.7}$$

Then, substituting for $\boldsymbol{y}(t)$ in Eqn. (3.5), we obtain

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}+\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{u}(t) \tag{3.8}$$

where $\boldsymbol{P}$ and $\boldsymbol{P}^{-1}$ matrices are

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ -\alpha & -\beta \end{bmatrix} \qquad \boldsymbol{P}^{-1} = \frac{1}{(\alpha-\beta)}\begin{bmatrix} -\beta & -1 \\ \alpha & 1 \end{bmatrix} \tag{3.9}$$

Eqn. (3.8) can be written in matrix form as

$$\dot{\boldsymbol{z}}(t) = \begin{bmatrix} -\alpha & 0 \\ 0 & -\beta \end{bmatrix}\boldsymbol{z}(t) + \begin{bmatrix} \dfrac{-1}{(\alpha-\beta)} \\ \dfrac{1}{\alpha-\beta} \end{bmatrix}\boldsymbol{u}(t) \tag{3.10}$$

or in scalar equations

$$\begin{aligned}\dot{z}_1(t) &= -\alpha z_1(t) - \frac{1}{(\alpha-\beta)}u(t)\\ \dot{z}_2(t) &= -\beta z_2(t) + \frac{1}{(\alpha-\beta)}u(t)\end{aligned} \tag{3.11}$$

For simplicity, we define the state variables $x_1(t)$ and $x_2(t)$ by the relations

$$\begin{aligned}x_1(t) &= \alpha(\alpha-\beta)z_1(t)\\ x_2(t) &= -\beta(\alpha-\beta)z_2(t)\end{aligned} \tag{3.12}$$

Then, $x_1(t)$ and $x_2(t)$ satisfy the differential equations

$$\begin{aligned}\dot{x}_1(t) &= -\alpha x_1(t) - \alpha u(t)\\ \dot{x}_2(t) &= -\beta x_2(t) - \beta u(t)\end{aligned} \tag{3.13}$$

or in matrix form

$$\dot{x}(t) = \begin{bmatrix}-\alpha & 0\\ 0 & -\beta\end{bmatrix}x(t) + \begin{bmatrix}-\alpha\\ -\beta\end{bmatrix}u(t) \tag{3.14}$$

Note that Eqn.(3.14) is of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.15}$$

Figure (3.1) illustrates in block diagram form, the linear transformation necessary to obtain $x_1(t)$ and $x_2(t)$ from $y_1(t)$ and $y_2(t)$. [Ref. 1]

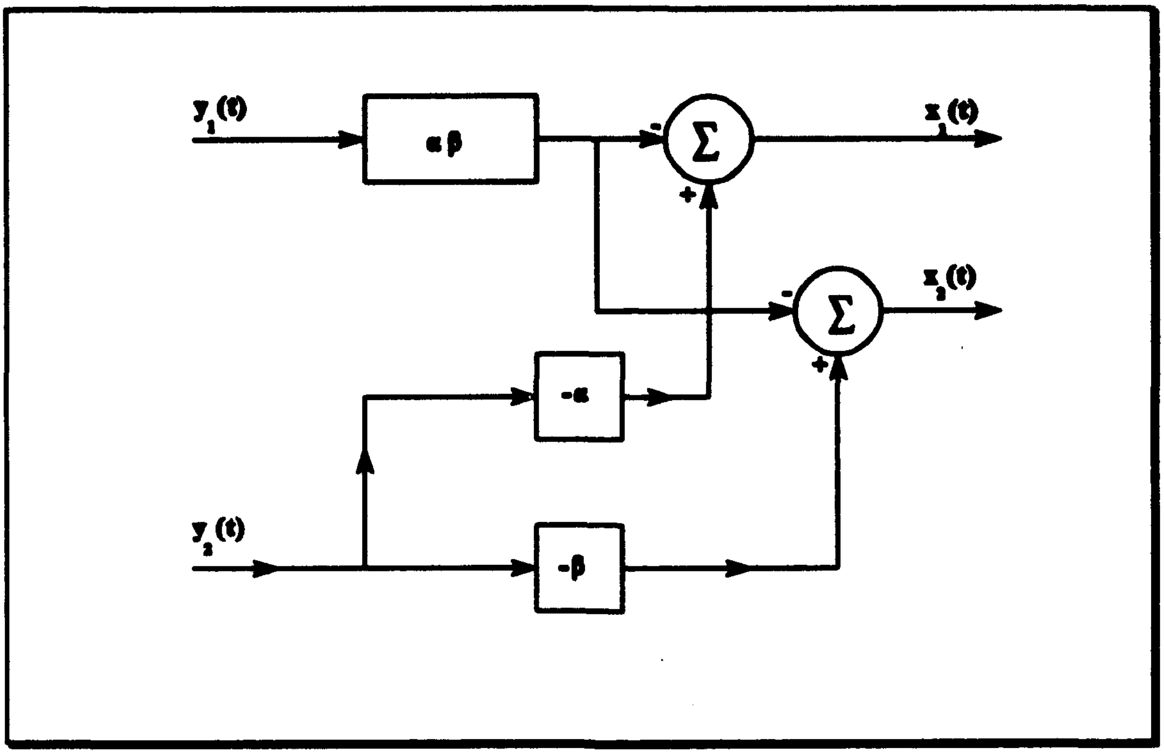

Figure 3.1 Block diagram of the linear transformation between x and y variables

We have thus transferred the original system (3.5) into an equivalent uncoupled system (3.15) using similarity transformations. Note that existence and uniqueness of time-optimal control holds also for the system (3.15).

2. Hamiltonian, H-Minimal Control, and the Equations of the Costate Variables

Let us write the Hamiltonian for this particular problem. We have

$$H = 1-\alpha x_1(t)p_1(t) - \beta x_2(t)p_2(t) + u(t)[-\alpha p_1(t) - \beta p_2(t)] \qquad (3.16)$$

Since the Hamiltonian H is linear in the control vector $u(t)$, minimization of the Hamiltonian with respect to $u(t)$ requires that [Ref. 3]

$$u(t) = sign\{\alpha p_1(t) + \beta p_2(t)\} \qquad (3.17)$$

where the costate variables $p_1(t)$ and $p_2(t)$ satisfy the differential equations

$$\begin{aligned}\dot{p}_1(t) &= -\frac{\partial H}{\partial x_1(t)} = \alpha p_1(t)\\ \dot{p}_2(t) &= -\frac{\partial H}{\partial x_2(t)} = \beta p_2(t)\end{aligned} \tag{3.18}$$

so that

$$\begin{aligned}p_1(t) &= e^{\alpha t} p_1(0)\\ p_2(t) &= e^{\beta t} p_2(0).\end{aligned} \tag{3.19}$$

Substituting Eqn. (3.19) into Eqn. (3.17), we find that

$$u(t) = sign\{\alpha p_1(0) e^{\alpha t} + \beta p_2(0) e^{\beta t}\} \tag{3.20}$$

where the function $q(t) = \alpha p_1(0) e^{\alpha t} + \beta p_2(0) e^{\beta t}$ has at most one zero. Therefore, we conclude that the four control sequences

$$\{+1\}, \{-1\}, \{+1, -1\}, \{-1, +1\} \tag{3.21}$$

are the only candidates for the time-optimal control of this system.

3. Equation of the Switching Curve

Since, the control must be piecewise constant, we solve Eqns.(3.13) using

$$u(t) = \Delta = \pm 1 \tag{3.22}$$

to obtain the solution

$$x_1(t) = (\xi_1+\Delta)e^{-\alpha t}-\Delta \qquad x_2(t) = (\xi_2+\Delta)e^{-\beta t}-\Delta \tag{3.23}$$

where $\xi_i = x_i(0)$, $i = 1,2$. Eliminating the time t in Eqs. (3.23) and setting

$$\varphi = \frac{\beta}{\alpha} \qquad (0<\alpha<\beta) \tag{3.24}$$

we find that

$$x_2(t) = -\Delta+(\xi_2+\Delta)\left(\frac{x_1(t)+\Delta}{\xi_1+\Delta}\right)^{\varphi} \tag{3.25}$$

Equation (3.25) describes a trajectory in the x_1x_2 plane. The trajectory originates at the state (ξ_1,ξ_2) and evolves as a result of the action of the constant control $u(t) = \Delta$. Since the eigenvalues are negative, then the trajectories generated by $u(t) = -1$, which we call *-1 forced trajectories*, tend to the state (1, 1) of the state plane. The trajectories generated by $u(t) = +1$ which we call *+1 forced trajectories*, tend to the state (-1, -1) of the state plane [Ref. 1]. Since,

$$\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} (\xi_i+\Delta)^{\lambda_i t}-\Delta = \pm 1 \qquad i=1,2 \tag{3.26}$$

where λ_i are the eigenvalues and $\Delta = \pm 1$. The *-1 forced trajectories* are shown in Fig.(3.2) and the *+1 forced trajectories* are shown in Fig.(3.3). Since the origin of the state space is the desired terminal state and since we must reach the origin using either control $u = +1$ or the control $u = -1$, we isolate the two forced trajectories which pass

through the origin. We denote these trajectories to the origin by γ_+ and γ_-. More precisely, γ_+ is given by

$$\gamma_+ = \left\{ (x_1, x_2) : -1 + (x_2 + 1)\left(\frac{1}{1 + x_1} \right)^{\varphi} = 0; \quad x_1 > 0,\ x_2 > 0 \right\} \tag{3.27}$$

The γ_- curve is given by

$$\gamma_- = \left\{ (x_1, x_2) : 1 + (x_2 - 1)\left(\frac{1}{1 - x_1} \right)^{\varphi} = 0; \quad x_1 < 0,\ x_2 < 0 \right\} \tag{3.28}$$

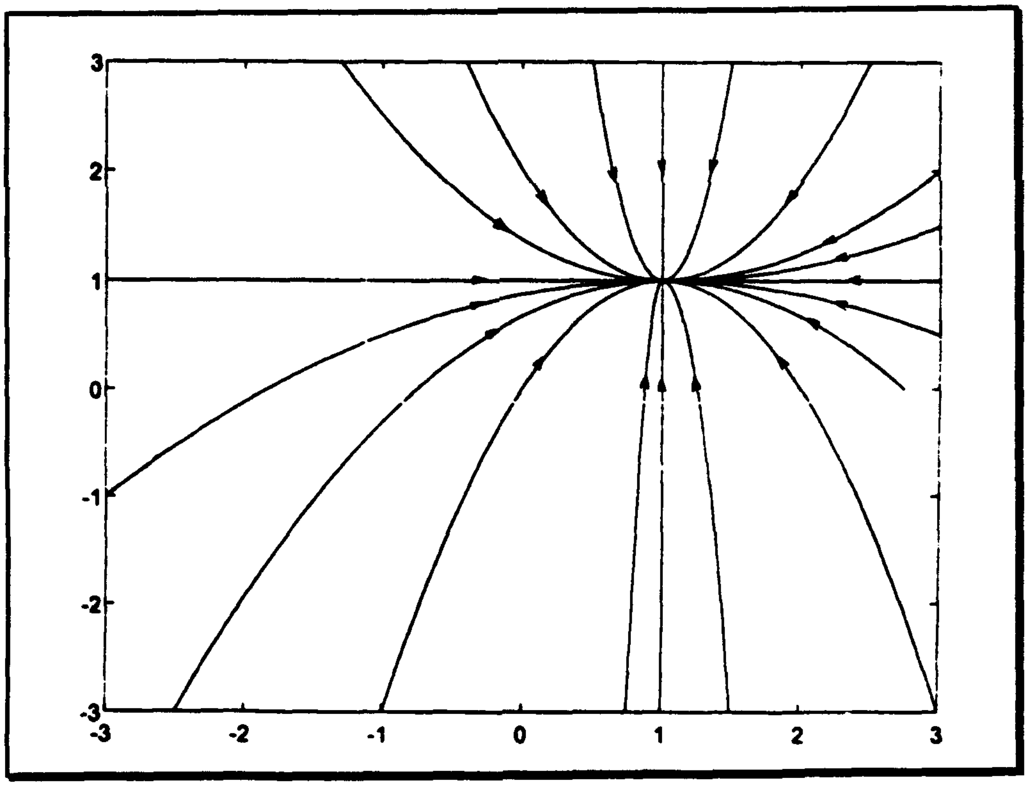

Figure 3.2 -1 Forced Trajectories

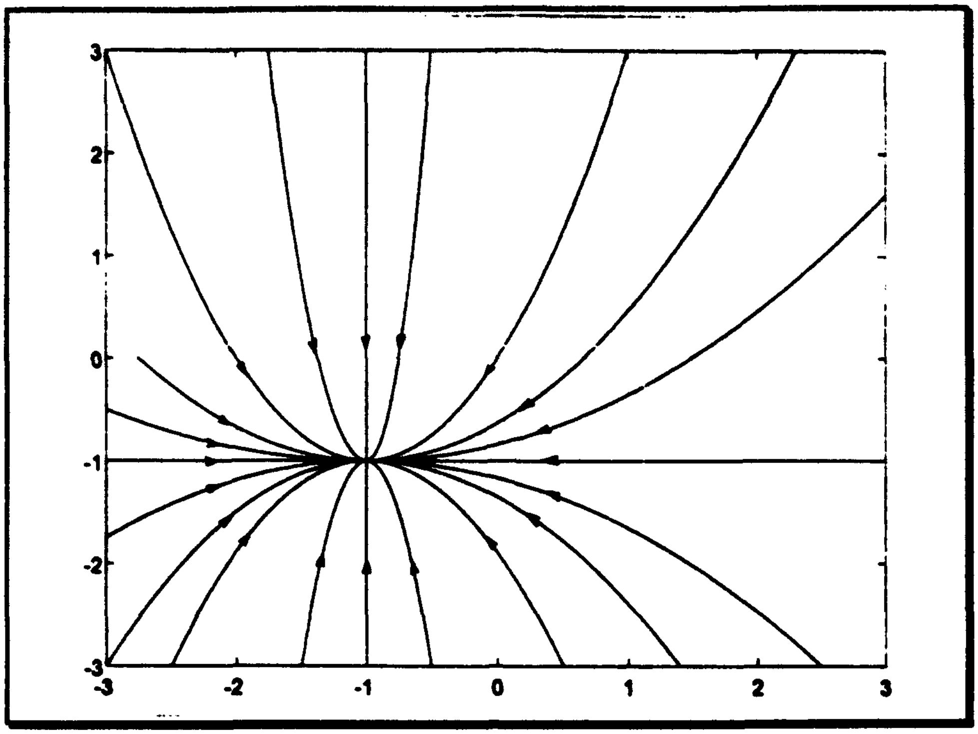

Figure 3.3 +1 Forced Trajectories

Using the shape of the forced trajectories shown in Figs. (3.2) and (3.3), we can conclude that only the control sequence { +1 } can force any state on the γ_+ curve to the origin. Similarly, only the control sequence { -1 } can force any state on the γ_- curve to the origin. Thus, we have derived the control law,

$$\text{If } (x_1,x_2) \in \gamma_+ \text{ then } u^*(t) = +1 \tag{3.29}$$

$$\text{If } (x_1,x_2) \in \gamma_- \text{ then } u^*(t) = -1 \tag{3.30}$$

We call the union of the γ_+ and the γ_- curves *switching curve* [Ref. 1]. Then combining Eqns. (3.27) and (3.28), the γ curve is given by

$$\gamma = \{ (x_1,x_2) : x_2 = \frac{x_1}{|x_1|}[(1+|x_1|)^q - 1] \} \tag{3.31}$$

Switching curves in $x_1 x_2$ plane for uncoupled system (3.15), and in $y_1 y_2$ plane for the system (3.5) are shown in Figs. (3.4) and (3.5).

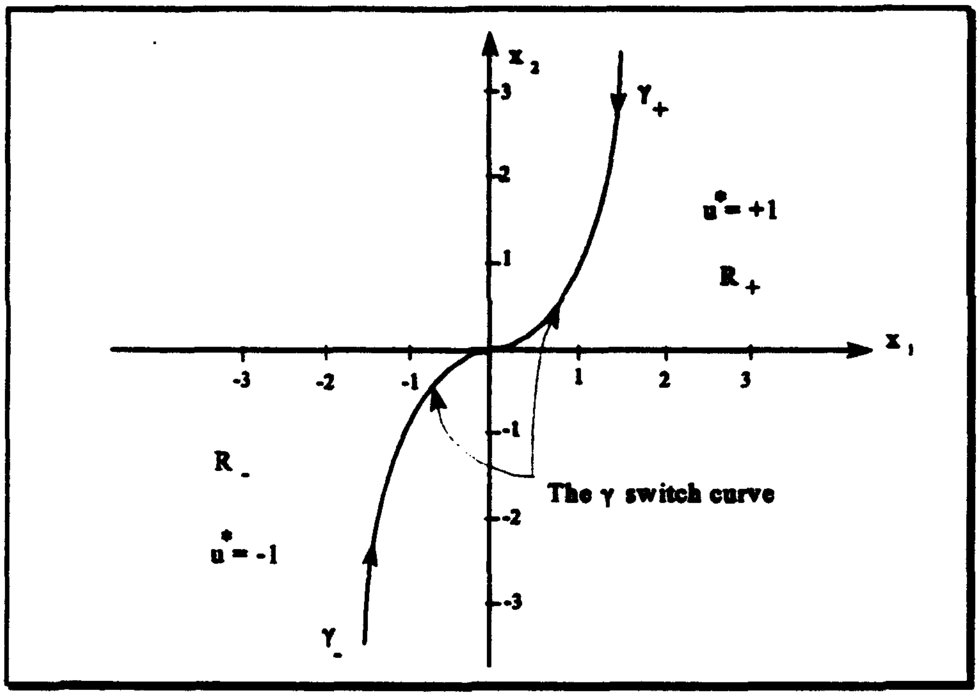

Figure 3.4 Switching Curve for the system (3.15)

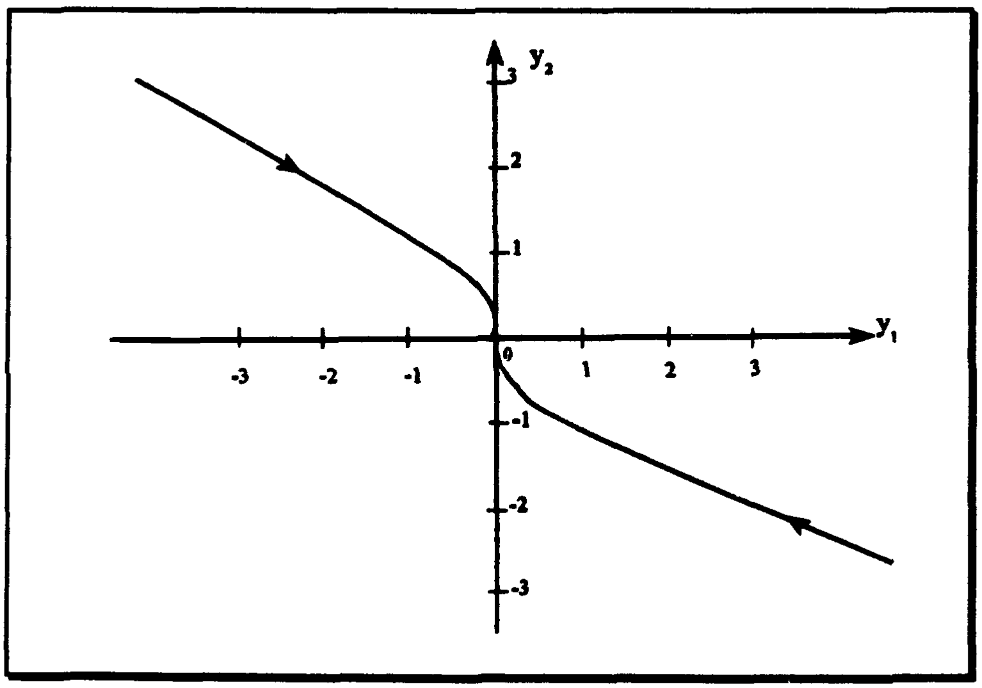

Figure 3.5 Switching Curve for the system (3.5)

Let us denote the set of states to the right of the γ curve as R_+, and the set of points to the left of the γ curve as R_-. Clearly,

$$
\begin{aligned}
R_+ &= \{(x_1,x_2) : x_2 < \frac{x_1}{|x_1|}[(1+|x_1|)^{\bullet}-1]\} \\
R_- &= \{(x_1,x_2) : x_2 > \frac{x_1}{|x_1|}[(1+|x_1|)^{\bullet}-1]\}
\end{aligned}
\tag{3.32}
$$

Using Eqns.(3.32), we can conclude that the control sequence { +1, -1 } can force any state belonging to set R_+ to the origin, and control sequence { -1, +1 } can force any state belonging to set R_- to the origin. Then, time-optimal control, as a function of the state (x_1,x_2) is given by

$$
\begin{aligned}
u^* = u^*(x_1,x_2) &= +1 \quad \forall \quad (x_1,x_2) \in \gamma_+ \cup R_+ \\
u^* = u^*(x_1,x_2) &= -1 \quad \forall \quad (x_1,x_2) \in \gamma_- \cup R_-
\end{aligned}
\tag{3.33}
$$

or in other words, the optimal control in terms of the state variables

$$
u^*(x_1,x_2) = sign\{\frac{x_1}{|x_1|}[(1+|x_1|)^{\bullet}-x_2-1]\}
\tag{3.34}
$$

An alternative solution to the optimal control in terms of states can be found in Ref. 4. We can easily obtain equations for switching curve and time-optimal control in terms of y_1 and y_2. We find that, $x_1(t)$ and $x_2(t)$ are related to $y_1(t)$ and $y_2(t)$ by

$$
\begin{aligned}
x_1(t) &= -\alpha\beta y_1(t) - \alpha y_2(t) \\
x_2(t) &= -\alpha\beta y_1(t) - \beta y_2(t)
\end{aligned}
\tag{3.35}
$$

The equations of the switching curve of time-optimal control are given by

$$
-\alpha\beta y_1(t) - \beta y_2(t) = \frac{-\alpha\beta y_1(t) - \alpha y_2(t)}{|-\alpha\beta y_1(t) - \alpha y_2(t)|}\{[1+|-\alpha\beta y_1(t) - \alpha y_2(t)|]^{\bullet}-1\}
\tag{3.36}
$$

and

$$u^*(y_1,y_2) = \mathrm{sign}\{\alpha\beta y_1(t) + \beta y_2(t) - \frac{\alpha\beta y_1(t) - \alpha y_2(t)}{|-\alpha\beta y_1(t) - \alpha y_2(t)|}\{[1 + |-\alpha\beta y_1(t) - \alpha y_2(t)|]^{\alpha} - 1\}\} \quad (3.37)$$

Equation (3.36) and (3.37) demonstrate the advantage of using $x_1(t)$ and $x_2(t)$ as state variables.

4. Analytic Solution for the Minimum Time t^*

We may now evaluate the minimum time t^* required to force any initial state (x_1,x_2) to the origin (0, 0) using the time-optimal control law given by Eqns.(3.33). Let us consider an initial state $X = (x_1,x_2)$, as shown in Fig. 3.6, and the time-optimal trajectory XWO to the origin, where $W = (w_1,w_2)$ is on the γ curve [Ref. 1].

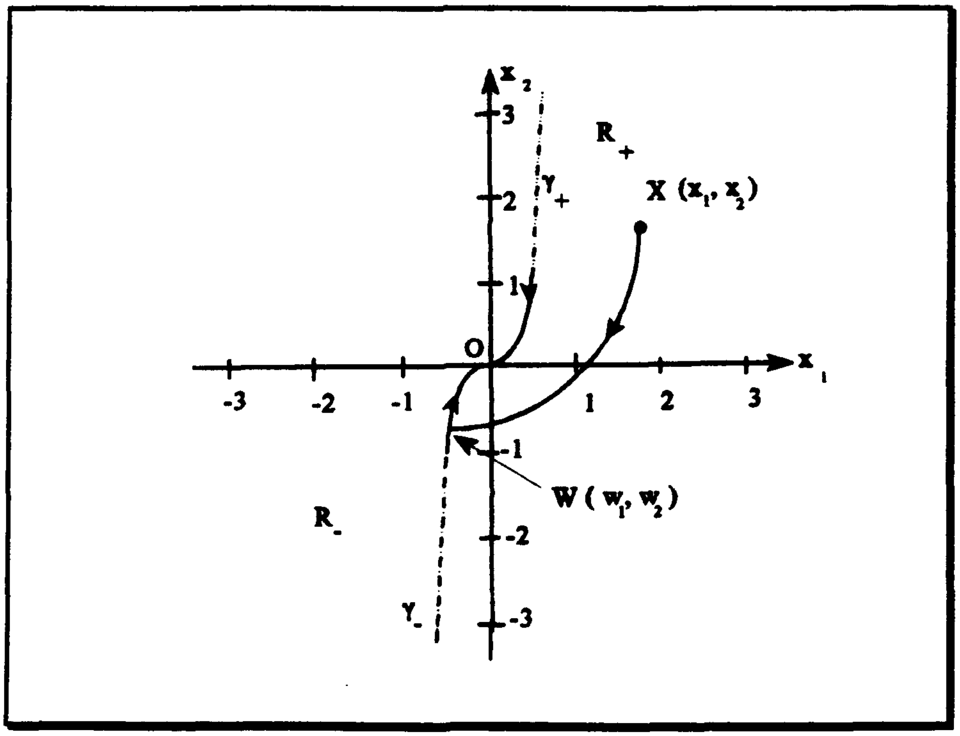

Figure 3.6 Time-optimal trajectory

Let us assume $u = \Delta^* = \pm 1$ is the optimal control applied during the trajectory WO, t_2 is the time required to go from W to O, and t_1 is the time required to go from X to W. Then, using Eqns.(3.23) we have

$$\begin{aligned} 0 &= (w_1 + \Delta^*)e^{-\alpha t_2} - \Delta^* \\ 0 &= (w_2 + \Delta^*)e^{-\beta t_2} - \Delta^* \end{aligned} \tag{3.38}$$

Solving for t_2 in the above equations, we find that

$$t_2 = -\frac{1}{\alpha}\log\left(\frac{\Delta^*}{w_1 + \Delta^*}\right) = -\frac{1}{\beta}\log\left(\frac{\Delta^*}{w_2 + \Delta^*}\right) \tag{3.39}$$

and from equation (3.38) we have

$$0 = -\Delta^* + (w_2 + \Delta^*)\left(\frac{\Delta^*}{w_1 + \Delta^*}\right)^{\phi} \tag{3.40}$$

where $\phi = \dfrac{\beta}{\alpha}$.

Using the shape of Fig. 3.6 and Eqns. (3.29) and (3.30), we conclude that,

$$\Delta^* = sign\{w_1\} = sign\{w_2\} \tag{3.41}$$

We again use Eqns. (3.23) to obtain t_1. Then, we have

$$\begin{aligned} w_1 &= (x_1 - \Delta^*)e^{-\alpha t_1} + \Delta^* \\ w_2 &= (x_2 - \Delta^*)e^{-\beta t_1} + \Delta^* \end{aligned} \tag{3.42}$$

Note that we use $u = -\Delta^*$. Since, during the trajectory XW we have $u = -\Delta^*$. Solving for t_1 in the above equation we obtain

$$t_1 = -\frac{1}{\alpha}\log\left(\frac{w_1 - \Delta^*}{x_1 - \Delta^*}\right) = -\frac{1}{\beta}\log\left(\frac{w_2 - \Delta^*}{x_2 - \Delta^*}\right) \tag{3.43}$$

and from Eqn.(3.43), we have

$$w_2 = \Delta^* + (x_2 - \Delta^*)\left(\frac{w_1 - \Delta^*}{x_1 - \Delta^*}\right)^{\phi} \tag{3.44}$$

Since, $t^* = t_1 + t_2$, then t^* is given by

$$t^* = -\frac{1}{\alpha}\log\left(\left(\frac{\Delta^*}{w_1 + \Delta^*}\right)\left(\frac{w_1 - \Delta^*}{x_1 - \Delta^*}\right)\right) \tag{3.45}$$

We want to find t^* as a function of x_1 and x_2 only, so we must eliminate w_1 from Eqn. (3.45). Combining Eqns.(3.40) and (3.44), we find that

$$0 = -\Delta^* + \left(2\Delta^* + (x_2 - \Delta^*)\left(\frac{w_1 - \Delta^*}{x_1 - \Delta^*}\right)^{\phi}\right)\left(\frac{\Delta^*}{w_1 + \Delta^*}\right)^{\phi} \tag{3.46}$$

which provides us a relationship between w_1, x_1, and x_2. For specific values of α and β Eqn.(3.46) reduces to a quadratic expression. For example for $\alpha = 1$ and $\beta = 2$, we have

$$w_1 = 4x_1 - 2x_2 - 2\Delta^* x_1^2 \pm \sqrt{(8\Delta^* x_2 - 16\Delta^* x_1 + 8x_1^2)(1 - \Delta^* x_1)^2} \tag{3.47}$$

Choosing the appropriate sign for w_1 and substituting into Eqn.(3.45), we can obtain an analytic expression in terms of the state $X = (x_1, x_2)$. Since we know the sign of Δ^* from Eqns.(3.33), we can easily obtain the switching time and the minimum time t^*, required to drive any initial state to the origin.

5. Simulation of the Linear Time-Optimal Regulator

Using a computer simulation, we test the accuracy of the solutions by choosing the initial conditions in the regions defined by Eqns.(3.32) with $\alpha = 1$ and $\beta = 2$. Figure 3.7 shows the state trajectories for the system (3.15) emanating from the region R_+. Time-optimal control as a function of time is shown in Fig. 3.8. As we claimed before, the control sequence **{ +1,-1}** drives the states to the origin with at most *n-1* switching in time-optimal control. State trajectories in $y_1 y_2$ plane are shown in Fig. 3.9. Next, we simulate the system (3.15) with initial conditions emanating from the region R_-. State trajectories and the time-optimal control are shown in Fig. 3.10 and Fig. 3.11. This time, the control sequence **{-1, +1}** drives the states to the origin with at most *n-1* change, in the control function as we suggested before. State trajectories in $y_1 y_2$ plane are shown in Fig. 3.12.

The desired terminal state was the origin of the state space. Upon reaching the origin the control effort must be shut off in order to maintain the system at rest. In both simulations switching time and minimum time t^* agree with the calculated values obtained from Eqns.(3.39), (3.43), (3.45), and, (3.47).

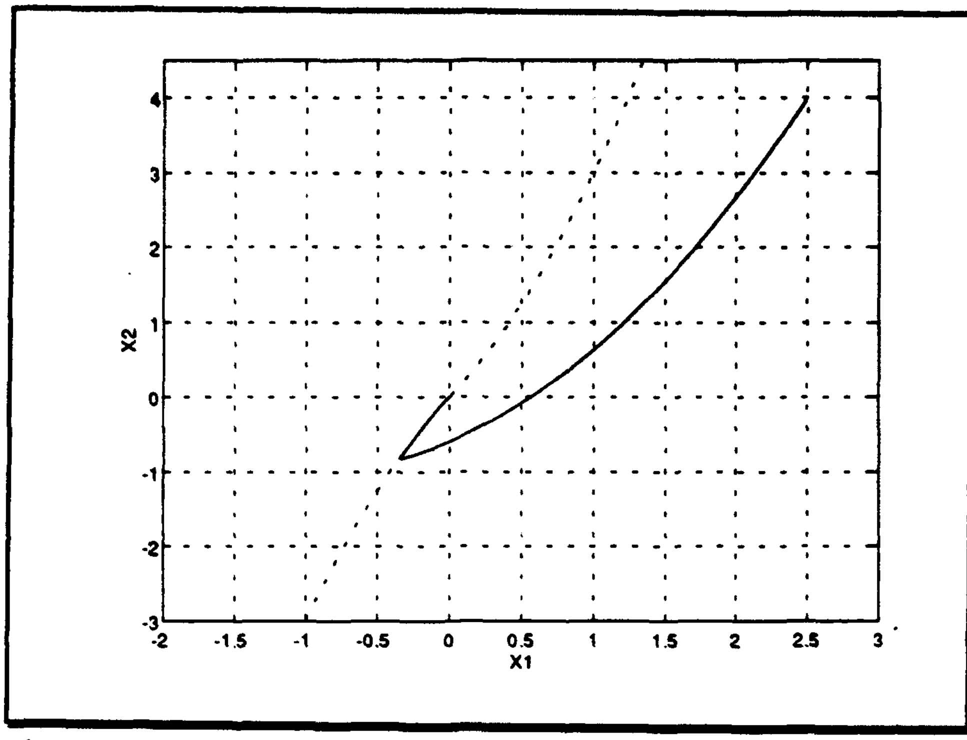

Figure 3.7 Optimal state trajectories in x_1x_2 plane

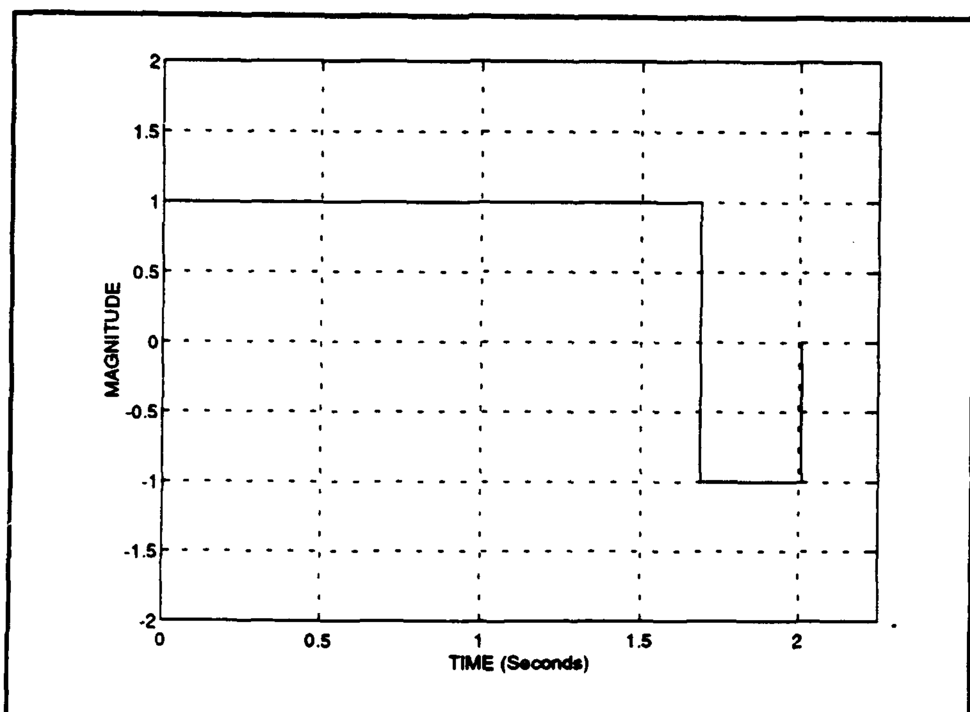

Figure 3.8 Time-optimal control

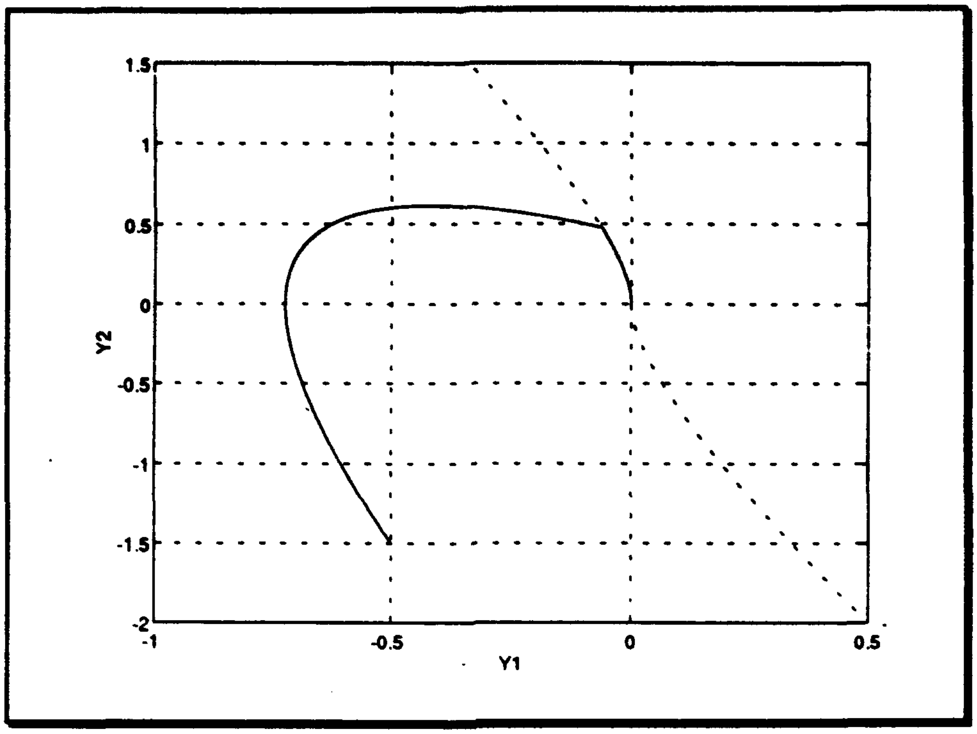

Figure 3.9 State trajectories in y_1y_2 plane

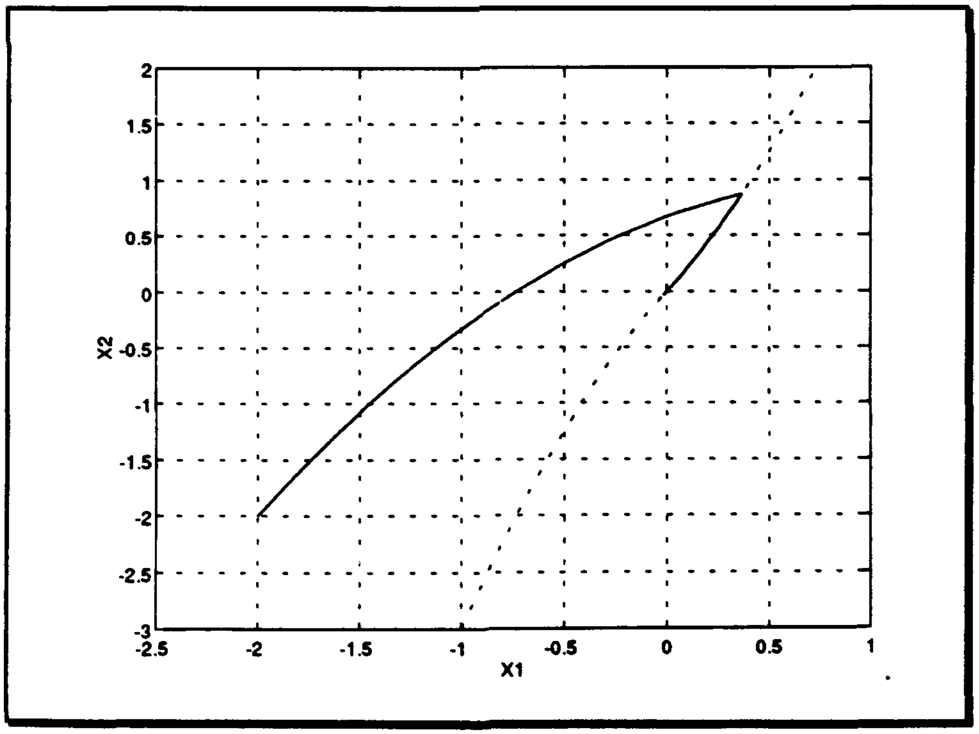

Figure 3.10 State trajectories in x_1x_2 plane

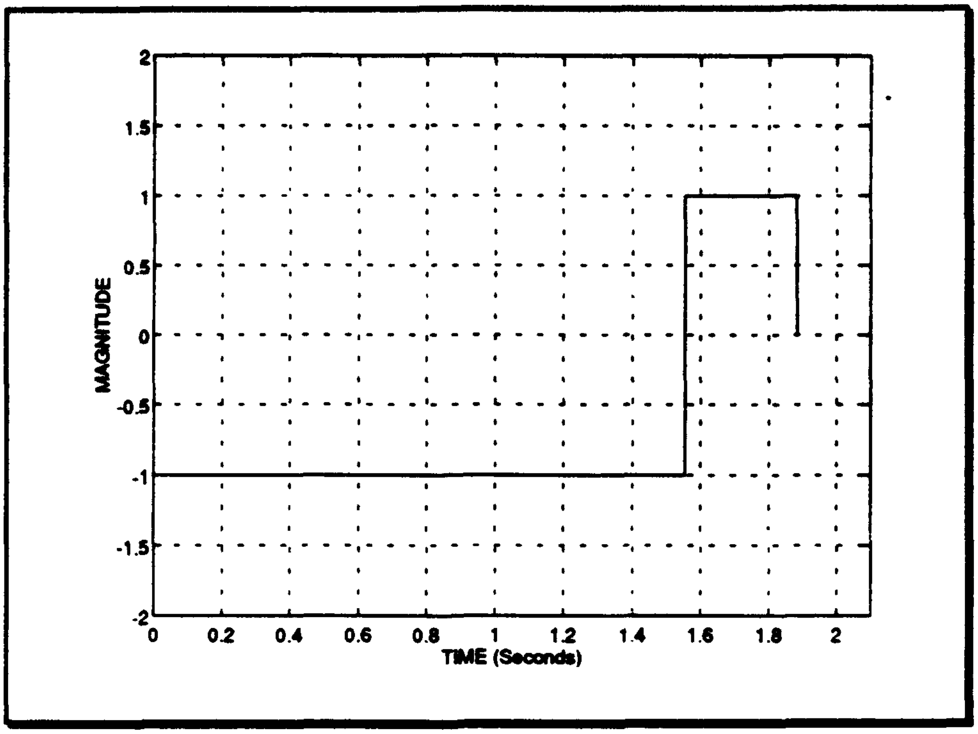

Figure 3.11 Time-optimal control

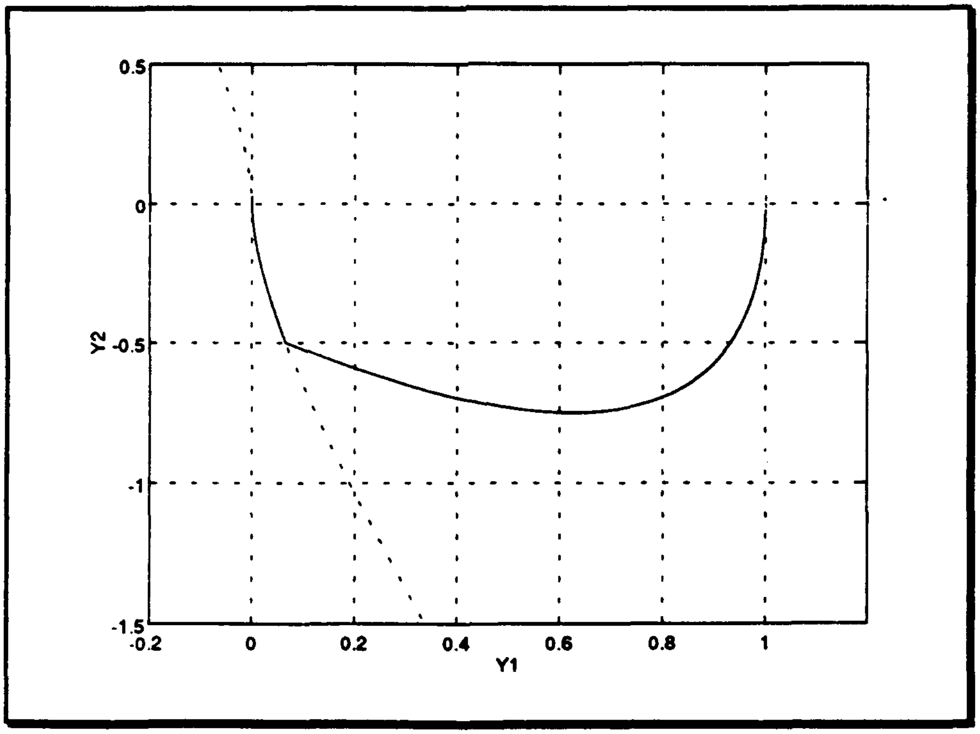

Figure 3.12 State trajectories in y_1y_2 plane

IV. TIME-OPTIMAL CONTROL OF A THIRD-ORDER PLANT WITH THREE REAL EIGENVALUES

A. GENERAL

In the previous chapter we solved the time-optimal control problem for a second-order plant. We showed that the time-optimal control can be determined as a function of the state by means of a switching curve which divides the state plane into two regions. In this chapter we consider the time-optimal control problem for a third-order plant with three distinct real and negative eigenvalues.

1. Problem Definition

We examine the system described by the third-order differential equation

$$\frac{d^3y(t)}{dt^3}+(\alpha+\beta+\gamma)\frac{d^2y(t)}{dt^2}+(\alpha\beta+\alpha\gamma+\gamma\beta)\frac{dy(t)}{dt}+(\alpha\beta\gamma)y(t) = u(t) \tag{4.1}$$

where α, β, and γ are real, distinct, nonzero eigenvalues. The transfer function of the system is

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{(s+\alpha)(s+\beta)(s+\gamma)} \tag{4.2}$$

Using Eqn.(4.2) the state space equations can be written in matrix form as

$$y(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha\beta\gamma & -(\alpha\beta+\alpha\gamma+\gamma\beta) & -(\alpha+\beta+\gamma) \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{4.3}$$

Again, as a first step we need to check the controllability and the observability of the system (4.3). Since Q and R matrices, given by Eqns.(2.17) and (2.18) are both nonsingular, the system is controllable and observable. The eigenvalues are all nonpositive and real, so an optimal control exists.

Using partial fraction expansion, we decouple the system (4.3) with P and P^{-1} given by

$$P = \begin{bmatrix} \dfrac{1}{(\beta-\alpha)(\gamma-\alpha)} & \dfrac{1}{(\alpha-\beta)(\gamma-\beta)} & \dfrac{1}{(\alpha-\gamma)(\beta-\gamma)} \\ \dfrac{\alpha}{(\alpha-\beta)(\gamma-\alpha)} & \dfrac{\beta}{(\alpha-\beta)(\beta-\gamma)} & \dfrac{\gamma}{(\alpha-\gamma)(\gamma-\beta)} \\ \dfrac{\alpha^2}{(\beta-\alpha)(\gamma-\alpha)} & \dfrac{\beta^2}{(\alpha-\beta)(\gamma-\beta)} & \dfrac{\gamma^2}{(\alpha-\gamma)(\beta-\gamma)} \end{bmatrix} \tag{4.4}$$

and

$$P^{-1} = \begin{bmatrix} \alpha\beta & \beta+\gamma & 1 \\ \alpha\gamma & \gamma+\alpha & 1 \\ \alpha\beta & \beta+\alpha & 1 \end{bmatrix} \tag{4.5}$$

As before we use Eqns.(3.7) and (3.8) to obtain

$$\dot{z}(t) = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & -\gamma \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \tag{4.6}$$

where $z(t)$ satisfies the differential equations

$$\begin{aligned} \dot{z}_1(t) &= -\alpha z_1(t) + u(t) \\ \dot{z}_2(t) &= -\beta z_2(t) + u(t) \\ \dot{z}_3(t) &= -\gamma z_3(t) + u(t) \end{aligned} \tag{4.7}$$

For simplicity, we define the uncoupled state variables $x_1(t)$, $x_2(t)$, and $x_3(t)$ as

$$
\begin{aligned}
x_1(t) &= \alpha z_1(t) \\
x_2(t) &= \beta z_2(t) \\
x_3(t) &= \gamma z_3(t)
\end{aligned}
\tag{4.8}
$$

where $x_1(t)$, $x_2(t)$, and $x_3(t)$ satisfy the differential equations

$$
\begin{aligned}
\dot{x}_1(t) &= -\alpha x_1(t) + \alpha u(t) \\
\dot{x}_2(t) &= -\beta x_2(t) + \beta u(t) \\
\dot{x}_3(t) &= -\gamma x_3(t) + \gamma u(t)
\end{aligned}
\tag{4.9}
$$

or in matrix form

$$
\dot{x}(t) = \begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & -\gamma \end{bmatrix} x(t) + \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} u(t) \tag{4.10}
$$

2. Hamiltonian, H-Minimal Control, and the Equations of the Costate Variables

The Hamiltonian is

$$
H = 1 - \alpha p_1(t)x_1(t) - \beta p_2(t)x_2(t) - \gamma p_3(t)x_3(t) + u(t)[\alpha p_1(t) + \beta p_2(t) + \gamma p_3(t)] \tag{4.11}
$$

The control $u(t)$ which minimizes the Hamiltonian is

$$
u(t) = -sign\{\alpha p_1(t) + \beta p_2(t) + \gamma p_3(t)\} \tag{4.12}
$$

where the costate variables $p_i(t)$, $i = 1,2,3$ satisfy the equations

$$\begin{aligned} \dot{p}_1(t) &= -\frac{\partial H}{\partial x_1(t)} = \alpha p_1(t) \\ \dot{p}_2(t) &= -\frac{\partial H}{\partial x_2(t)} = \beta p_2(t) \\ \dot{p}_3(t) &= -\frac{\partial H}{\partial x_3(t)} = \gamma p_3(t) \end{aligned} \tag{4.13}$$

so that

$$\begin{aligned} p_1(t) &= p_1(0)e^{\alpha t} \\ p_2(t) &= p_2(0)e^{\beta t} \\ p_3(t) &= p_3(0)e^{\gamma t} \end{aligned} \tag{4.14}$$

Substituting Eqn.(4.14) into Eqn.(4.12), we find that

$$u(t) = -\,sign\,\{\alpha p_1(0)e^{\alpha t} + \beta p_2(0)e^{\beta t} + \gamma p_3(0)e^{\gamma t}\} \tag{4.15}$$

Then, the candidates for time-optimal control are the six control sequences

$$\{+1\}, \{-1\}, \{+1,-1\}, \{-1,+1\}, \{+1,-1,+1\}, \{-1\ +1,-1\} \tag{4.16}$$

3. Equations of the Switching Curve and the Switching Surface

Again, we solve Eqns.(4.9) using

$$u(t) = \Delta = \pm 1 \tag{4.17}$$

to obtain the solutions

$$\begin{aligned} x_1(t) &= (\xi_1 - \Delta)e^{-\alpha t} + \Delta \\ x_2(t) &= (\xi_2 - \Delta)e^{-\beta t} + \Delta \\ x_3(t) &= (\xi_3 - \Delta)e^{-\gamma t} + \Delta \end{aligned} \tag{4.18}$$

Eliminating time t in Eqns.(4.18), we find that

$$
\begin{aligned}
x_2(t) &= (\xi_2 - \Delta)\left(\frac{x_1 - \Delta}{\xi_1 - \Delta}\right)^{\frac{\beta}{\alpha}} + \Delta \\
x_3(t) &= (\xi_3 - \Delta)\left(\frac{x_1 - \Delta}{\xi_1 - \Delta}\right)^{\frac{\gamma}{\alpha}} + \Delta
\end{aligned}
\tag{4.19}
$$

Eqn.(4.19) describes a trajectory in the three-dimensional state space. From Eqn.(4.18) we conclude that

$$
\lim_{t \to \infty} x_i(t) = \Delta , \quad i = 1,2,3 \tag{4.20}
$$

where $\Delta = \pm 1$. This means that a trajectory generated by $u = +1$ tends to the point (1,1,1) and a trajectory generated by $u = -1$ tends to the point (-1,-1,-1).

Now, let $\{ V_1 \}$ denote the set of states which can be forced to the origin (0,0,0) of the state space by application of the control $u = \Delta^* = \pm 1$. We use x_2 to indicate a state belonging to the set $\{ V_2 \}$ and $x_{1,2}$, $x_{2,2}$, $x_{3,2}$ to indicate the components of x_2. If t_2 denotes the positive time required to force x_2 to the origin using $u = \Delta^* = \pm 1$, then from Eqn.(4.18) we have

$$
\begin{aligned}
0 &= (x_{1,2} - \Delta^*) e^{-\alpha t_2} + \Delta^* \\
0 &= (x_{2,2} - \Delta^*) e^{-\beta t_2} + \Delta^* \\
0 &= (x_{3,2} - \Delta^*) e^{-\gamma t_2} + \Delta^*
\end{aligned}
\tag{4.21}
$$

or equivalently

$$
\begin{aligned}
x_{1,2} &= \Delta^* - \Delta^* e^{\alpha t_2} \\
x_{2,2} &= \Delta^* - \Delta^* e^{\beta t_2} \\
x_{3,2} &= \Delta^* - \Delta^* e^{\gamma t_2}
\end{aligned}
\tag{4.22}
$$

Let $\{V_1\}$ denote the set of states which can be forced to the set $\{V_2\}$ by application of the control $u = -\Delta^*$. Again, let vector x_1 indicate the states belonging to set $\{V_1\}$ and the components of x_1 be $x_{1,1}$, $x_{2,1}$, and $x_{3,1}$. If t_1 denotes the positive time required to force x_1 to a state $x_2 \in \{V_2\}$, then using Eqn.(4.18) we obtain the states belonging to $\{V_1\}$.

$$\begin{aligned} x_{1,2} &= (x_{1,1}+\Delta^*)e^{-\alpha t_1}-\Delta^* \\ x_{2,2} &= (x_{2,1}+\Delta^*)e^{-\beta t_1}-\Delta^* \\ x_{3,2} &= (x_{3,1}+\Delta^*)e^{-\gamma t_1}-\Delta^* \end{aligned} \tag{4.23}$$

or equivalently

$$\begin{aligned} x_{1,1} &= -\Delta^*+(x_{1,2}+\Delta^*)e^{\alpha t_1} \\ x_{2,1} &= -\Delta^*+(x_{2,2}+\Delta^*)e^{\beta t_1} \\ x_{3,1} &= -\Delta^*+(x_{3,2}+\Delta^*)e^{\gamma t_1} \end{aligned} \tag{4.24}$$

Thus, we have defined the sets $\{V_1\}$ and $\{V_2\}$. Eqns. (4.22) and (4.24) imply that the trajectory originating at any point $x_1 \in \{V_1\}$ and generated by $u = -\Delta^*$ will remain on the surface $\{V_1\}$ until it hits a point on the set (curve) $\{V_2\}$.

To simplify the Eqns. (4.22) and (4.24), we define new variables z_1, and z_2 by

$$\begin{aligned} z_1 &= e^{t_1} \\ z_2 &= e^{t_2} \end{aligned} \tag{4.25}$$

Using these new variables, Eqns. (4.22) and (4.24) become

$$\begin{aligned} x_{1,2} &= \Delta^*-\Delta^* z_2^{\alpha} \\ x_{2,2} &= \Delta^*-\Delta^* z_2^{\beta} \\ x_{3,2} &= \Delta^*-\Delta^* z_2^{\gamma} \end{aligned} \tag{4.26}$$

$$
\begin{aligned}
x_{1,1} &= -\Delta^{\circ} + (x_{1,2} + \Delta^{\circ}) z_1^{\alpha} \\
x_{2,1} &= -\Delta^{\circ} + (x_{2,2} + \Delta^{\circ}) z_1^{\beta} \\
x_{3,1} &= -\Delta^{\circ} + (x_{3,2} + \Delta^{\circ}) z_1^{\gamma}
\end{aligned}
\tag{4.27}
$$

Substituting Eqns. (4.26) in Eqns. (4.27) we obtain

$$
\begin{aligned}
\Delta^{\circ} x_{1,1} &= -1 + 2 z_1^{\alpha} - (z_1 z_2)^{\alpha} \\
\Delta^{\circ} x_{2,1} &= -1 + 2 z_1^{\beta} - (z_1 z_2)^{\beta} \\
\Delta^{\circ} x_{3,1} &= -1 + 2 z_1^{\gamma} - (z_1 z_2)^{\gamma}
\end{aligned}
\tag{4.28}
$$

For simplicity, we define new variables w_1, and w_2 by

$$
\begin{aligned}
w_1 &= z_1 \\
w_2 &= z_1 z_2
\end{aligned}
\tag{4.29}
$$

Since we specified the times t_1 and t_2 positive, this implies that

$$
1 < z_i \qquad\qquad i = 1,2 \tag{4.30}
$$

where z_i is given by Eqn.(4.25). Combining Eqns. (4.29) and (4.30), we obtain the inequality

$$
1 < w_1 < w_2 \tag{4.31}
$$

Using the variables w_1 , and w_2 , Eqns. (4.26) and (4.28) become

$$
\begin{aligned}
\Delta^{\circ} x_{1,2} &= 1 + 2 w_1^{\alpha} - w_2^{\alpha} \\
\Delta^{\circ} x_{2,2} &= 1 + 2 w_1^{\beta} - w_2^{\beta} \\
\Delta^{\circ} x_{3,2} &= 1 + 2 w_1^{\gamma} - w_2^{\gamma}
\end{aligned}
\tag{4.32}
$$

$$
\begin{aligned}
\Delta^{\circ} x_{1,1} &= -1 + 2 w_1^{\alpha} - w_2^{\alpha} \\
\Delta^{\circ} x_{2,1} &= -1 + 2 w_1^{\beta} - w_2^{\beta} \\
\Delta^{\circ} x_{3,1} &= -1 + 2 w_1^{\gamma} - w_2^{\gamma}
\end{aligned}
\tag{4.33}
$$

Now, let us state some important properties of the sets $\{ V_1 \}$ and $\{ V_2 \}$.

- The sets $\{ V_1 \}$ and $\{ V_2 \}$ are symmetric about the origin.
- In the three-dimensional state space, the set $\{ V_2 \}$ is a curve and $\{ V_1 \}$ is a surface. $\{ V_1 \}$ divides the state space into two parts.
- The sets $\{ V_1 \}$ and $\{ V_2 \}$ are formed by families of smooth and continuous trajectories.
- The sets $\{ V_1 \}$ and $\{ V_2 \}$ are infinite in extent. The origin is contained in the set $\{V_2\}$, the set $\{ V_2 \}$ is contained in $\{ V_1 \}$.
- The state (1,1,1) is above the surface $\{ V_1 \}$, and the state (-1,-1,-1) is below the surface $\{ V_1 \}$.

In order to determine the optimal control law, we need to find whether the state $\boldsymbol{x}$ given by

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{4.34}$$

is above, on, or below the surface $\{ V_1 \}$. We set

$$x_{1,1} = x_1, \qquad x_{2,1} = x_2 \tag{4.35}$$

in the first two Eqns. of (4.33) and solve these two equations to determine the values of w_1, w_2 and $\Delta^* = +1$ or $\Delta^* = -1$. We need to satisfy the inequality given by (4.31). Then, we substitute the values of w_1 , w_2, and Δ^* in the last equation of (4.33) and evaluate $x_{3,1}$. We can compare the computed value of $x_{3,1}$ with the last component of the state $\boldsymbol{x}$.

If

$$x_3 - x_{3,1} > 0 \tag{4.36}$$

then we say that x is above the surface $\{V_1\}$. If

$$x_3 - x_{3,1} = 0 \tag{4.37}$$

then $x \in \{V_1\}$, and if

$$x_3 - x_{3,1} < 0 \tag{4.38}$$

then we say that x is below the surface $\{V_1\}$. Figure 4.1 shows an illustration of the projection x_1, of a state x on the surface $\{V_1\}$ [Ref. 1]. We draw a straight line parallel to the x_3 axis through the point $x = (x_1, x_2, x_3)$ which intersects the surface $\{V_1\}$ at a point $x_1 = (x_1, x_2, x_{3,1})$. Comparison of x_3 with $x_{3,1}$ indicates whether x is above, on or below the surface.

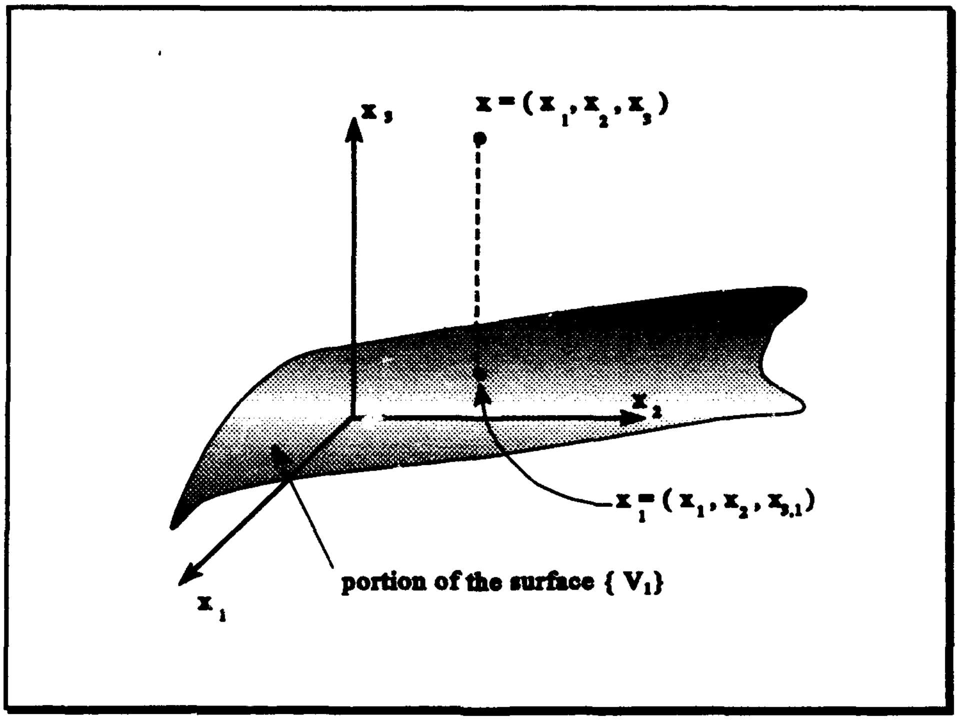

Figure 4.1 Projection x_1 of the state x on the surface $\{V_1\}$

4. Optimal Control u^*

Time-optimal control u^*, which forces any state x to the origin can be defined in the following way

$$
\begin{aligned}
&\text{If } x \text{ is above } \{V_1\} \text{ then } && u^* = (-1)^n \\
&\text{If } x \text{ is below } \{V_1\} \text{ then } && u^* = -(-1)^n \\
&\text{If } x \text{ is } \in \{V_1\} \text{ then } && u^* = (-1)^n \Delta^* \\
&\text{If } x \text{ is } \in \{V_2\} \text{ then } && u^* = -(-1)^n \Delta^*
\end{aligned}
\tag{4.39}
$$

Let us show that the control law given by Eqns. (4.39) is the time-optimal one. We recall that if the state x belongs to $\{V_1\}$, then the control switches exactly once. Since the system has three real eigenvalues, the time-optimal control can switch at most $n-1$ times. Let us consider the state

$$
x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{4.40}
$$

which is above $\{V_1\}$. Since n = 3, the control law (4.39) states that $u^* = -1$. Suppose that at x_1, we apply the control $u^* = +1$. From Eqn.(4.20), all the trajectories generated by $u^* = +1$ tend to the state x_1. So, the state will remain at x_1 forever. Therefore, to generate a trajectory which hits the surface $\{V_1\}$, we must apply $u^* = -1$ at x_1. If the control switches from $u = -1$ to $u = +1$ at $\{V_1\}$, the total number of switchings is $n-1$. This does not violate the necessary conditions. Now, let us consider the other states x which are above $\{V_1\}$. If $u = +1$, then x tends to the state (1,1,1), and eventually we must switch to $u = -1$ to reach $\{V_1\}$. But this method requires n switchings which violates the necessary

conditions. Therefore, if the state is above $\{V_1\}$ the control must be $u = -1$. By the same reasoning if the state is below $\{V_1\}$, the control $u = +1$ forces x to $\{V_1\}$. In either case, the total number of switchings is exactly n-1. [Ref. 1]

Now, we need to show that if a state x is on $\{V_1\}$ but not in $\{V_2\}$, the control must be $u = (-1)^n\Delta^*$. Suppose that for the point $x_1 \in \{V_1\}$, the value of Δ^* is -1. According to the control law (4.39), we must use $u = +1$ at x_1. Application of $u = +1$ generates a trajectory which follows the surface $\{V_1\}$ and hits $\{V_2\}$ at a point at which the control must switch to $u = -1$. This control sequence requires exactly one switching. Suppose that at x_1, we apply $u = -1$. The resulting trajectory will not follow the surface $\{V_1\}$. It will go below $\{V_1\}$, because, it will tend to the state (-1,-1,-1), which is below $\{V_1\}$ by definition. The control must switch from $u = -1$ to $u = +1$ so that the state is brought back to $\{V_1\}$. But this control sequence requires exactly n switchings. So it can not be a time-optimal one. From the above considerations, we conclude that the control which requires the minimum number of switchings is the time-optimal one. [Ref. 1]

5. Simulation of Minimum Time Control of the Third Order Regulator

We simulate system (4.10), with initial conditions above, below, and on the switching surface, using $\alpha = 1$, $\beta = 2$, and $\gamma = 3$. Figure 4.2 shows the three-dimensional state trajectories emanating from an initial point above the switching surface. Time-optimal control is shown in Fig. 4.3. The control sequence { -1, +1, -1 } drives the states to the origin with exactly two switchings as we suggested before. State trajectories as a function of time are shown in Fig. 4.4.

Next, we simulate the system starting from an initial condition below the switching surface. This time control sequence { +1, -1, +1 } drives the states to the origin, again with exactly two switchings. Three-dimensional state trajectories are shown in Fig. 4.5. Time-optimal control and the state trajectories as a function of time are shown in Figs. 4.6 and 4.7.

Lastly, we simulate the system starting from a point on the switching surface. Figure 4.8 shows the three-dimensional state trajectories. Time-optimal control and the state trajectories as a function of time are shown in Figs. 4.9 and 4.10. From Fig. 4.9 the control law { -1, +1} drives the states to the origin with only one switching .

All three simulations confirm that the control law given by (4.39) is the time-optimal control. Again, upon reaching the origin the control effort must be shut off to keep the system at rest.

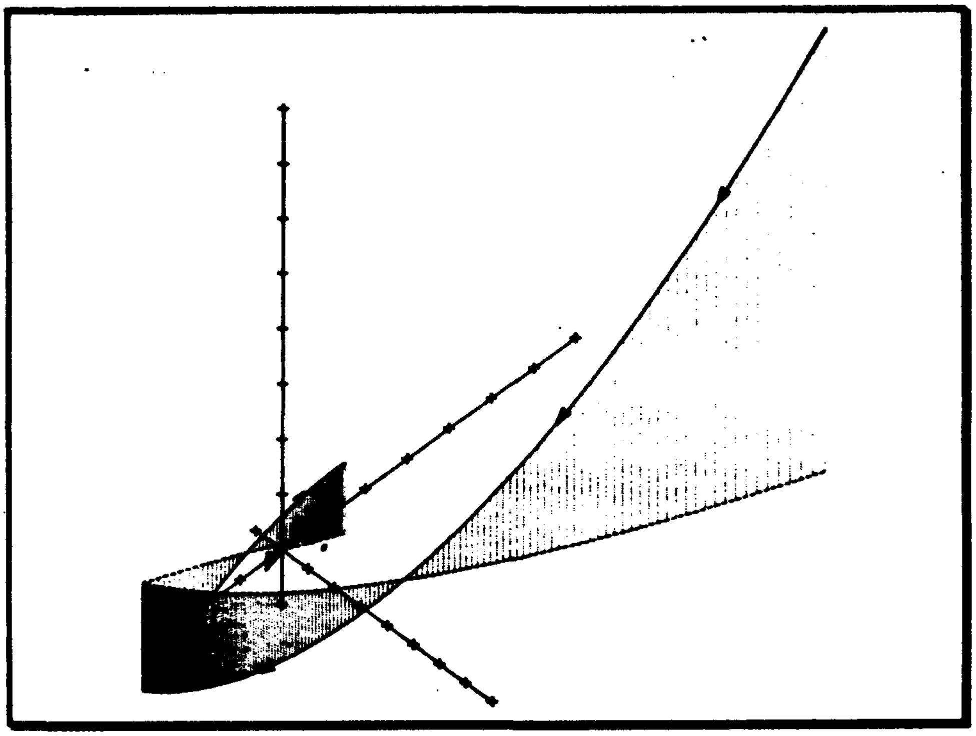

Figure 4.2 Three-dimensional state trajectories of system (4.10)

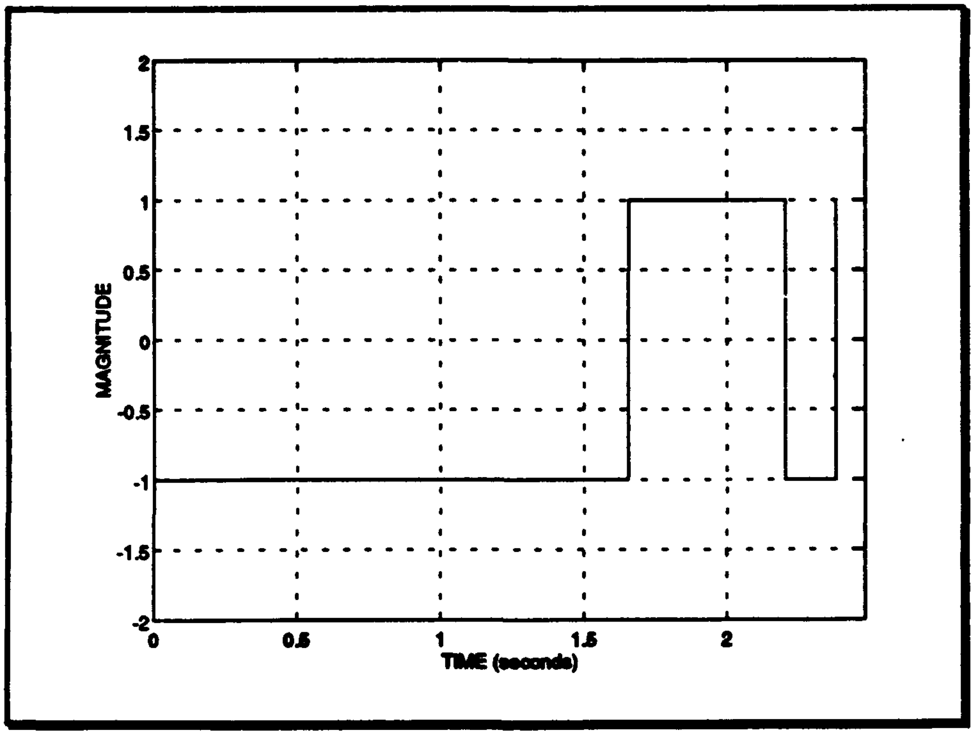

Figure 4.3 Time-optimal control

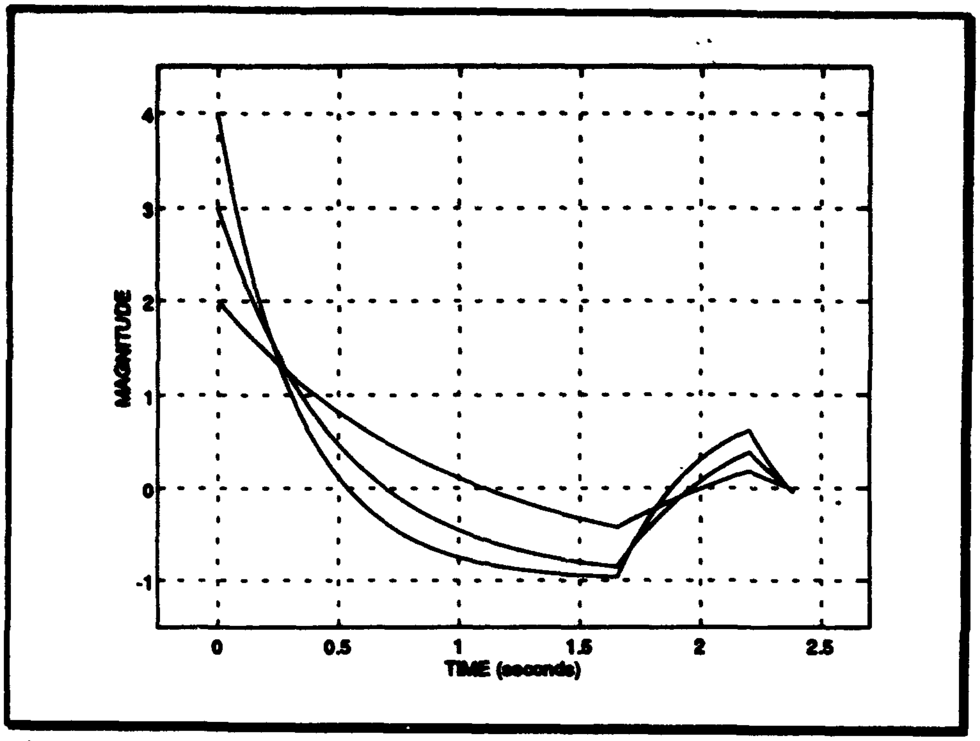

Figure 4.4 State trajectories as a function of time

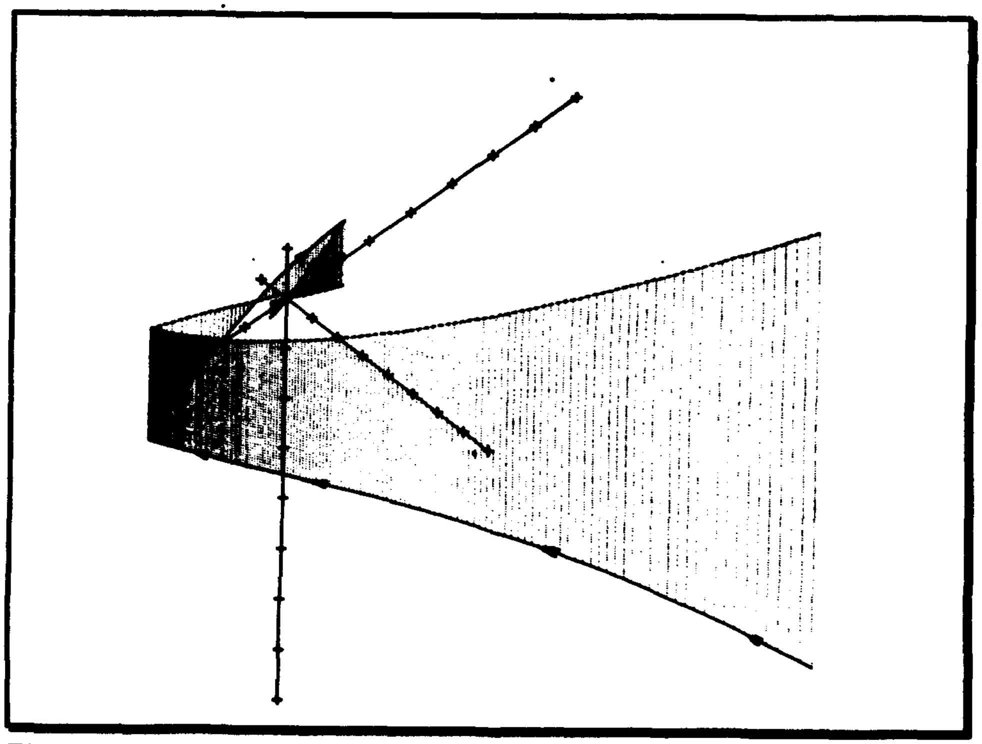

Figure 4.5 Three-dimensional state trajectories of system (4.10)

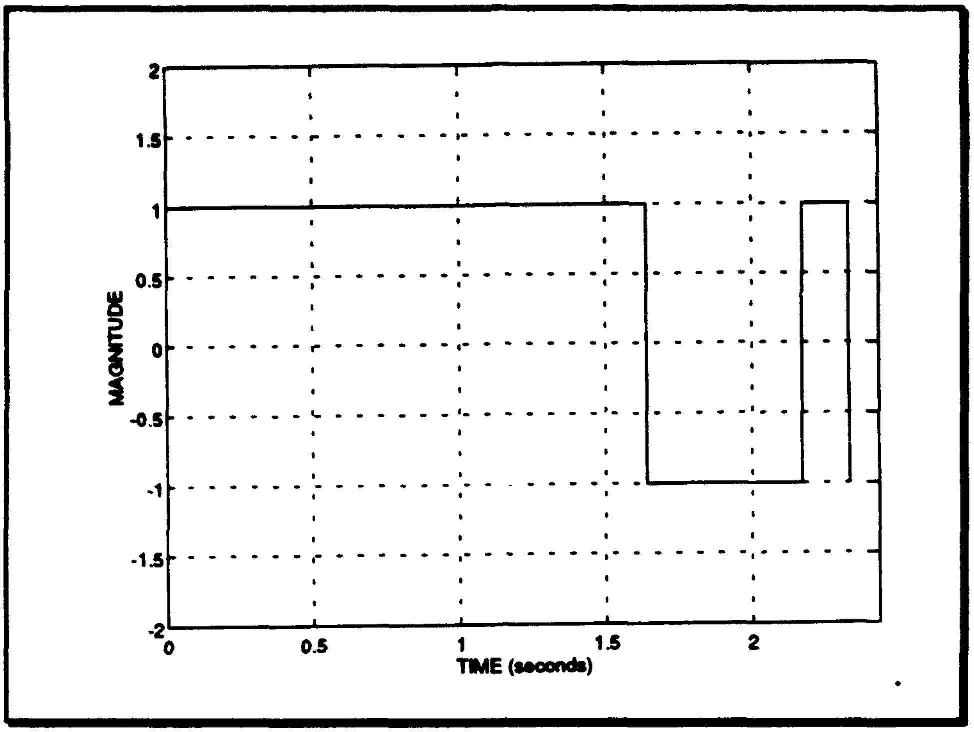

Figure 4.6 Time-optimal control

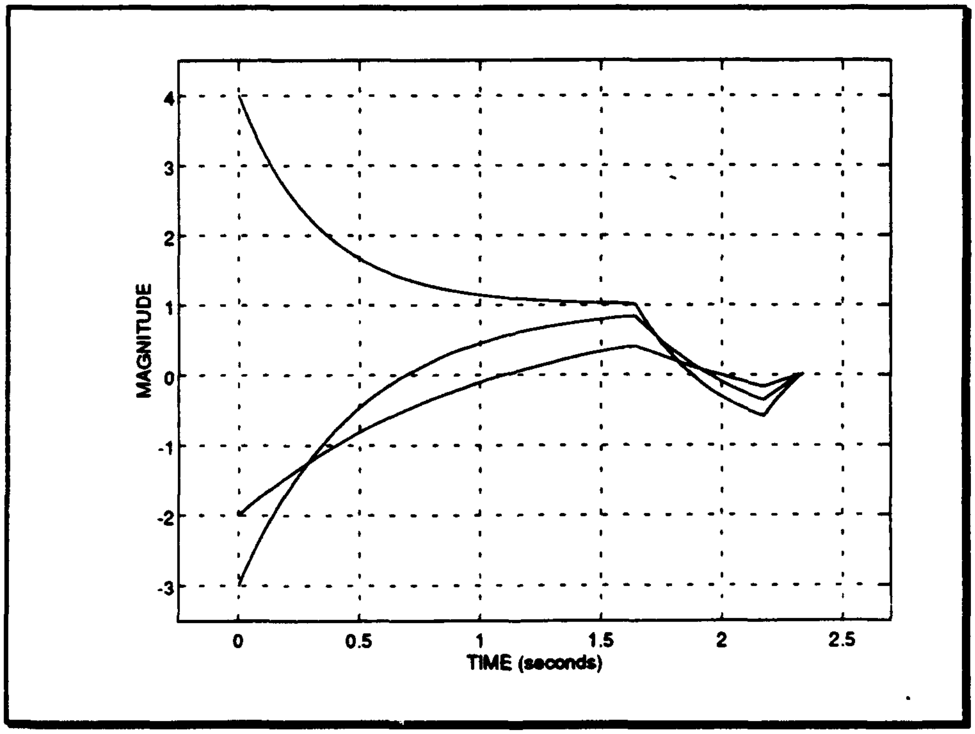

Figure 4.7 State trajectories as a function of time

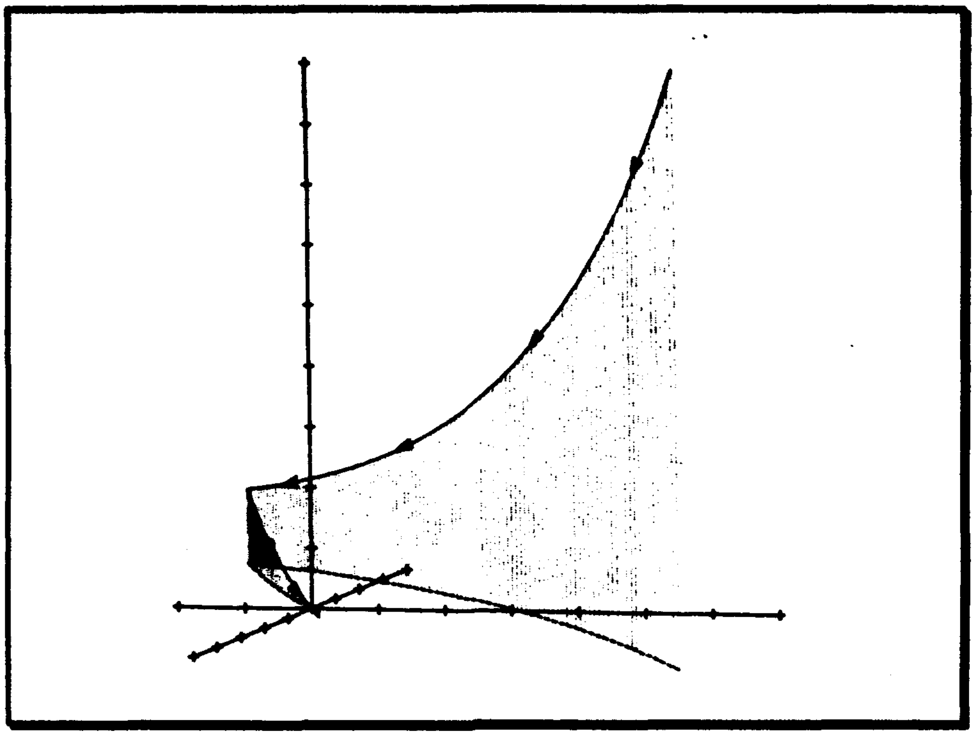

Figure 4.8 Three-dimensional state trajectories of system (4.10)

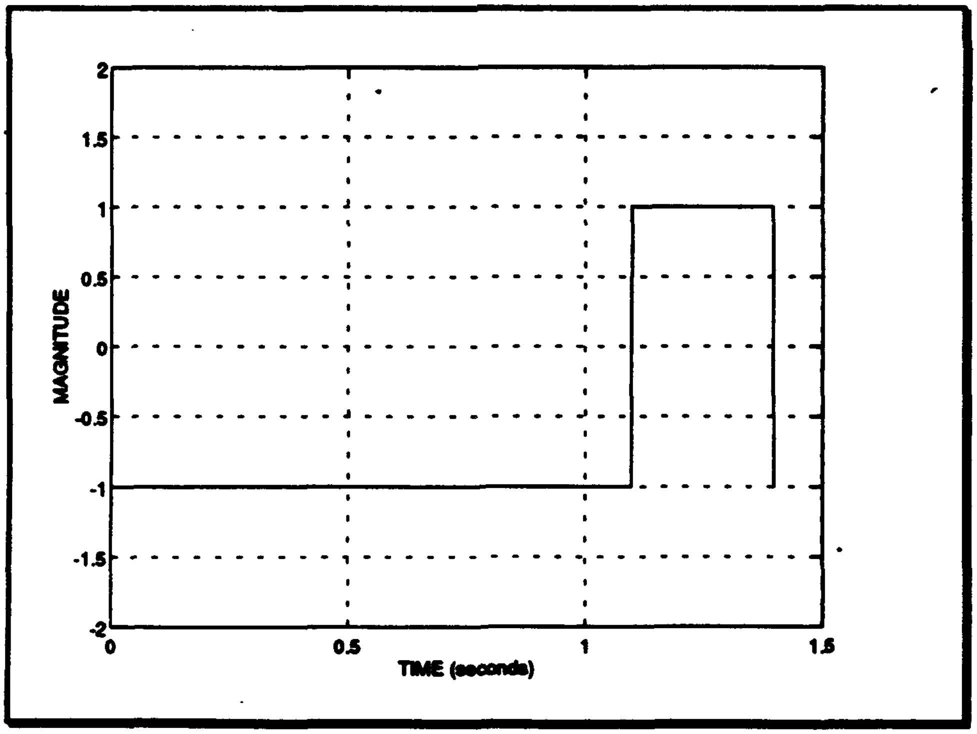

Figure 4.9 Time-optimal control

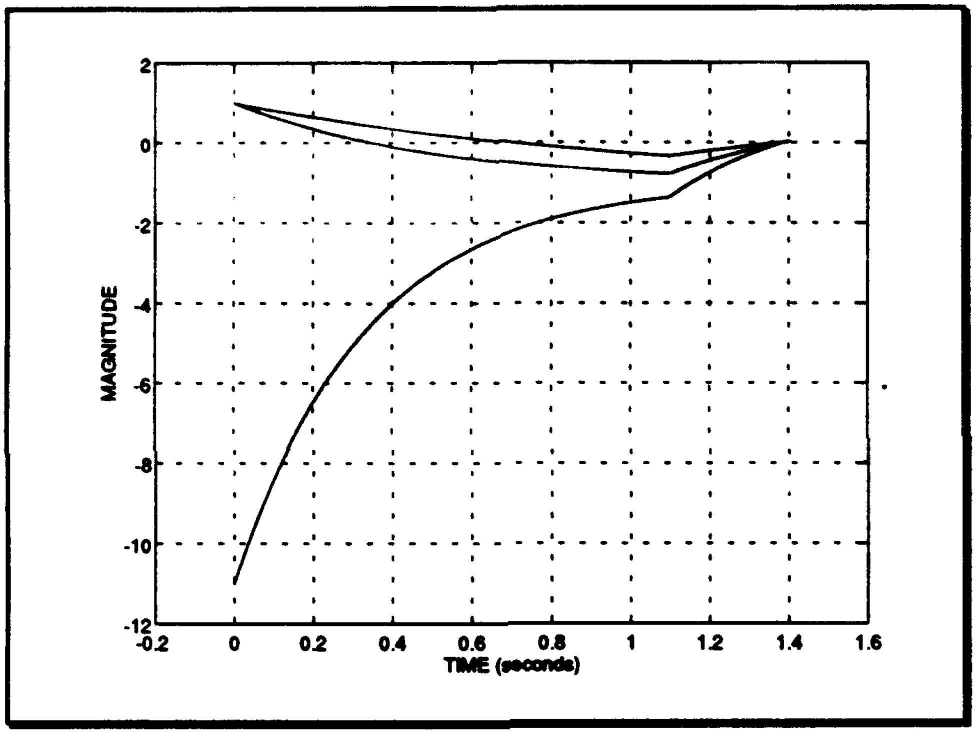

Figure 4.10 State trajectories as a function of time

V. CONCLUSION

We have examined the time-optimal control problem for one second-order and one third-order system. These systems had the following properties in common:

- The systems were linear, and time invariant.
- The transfer function of the system did not contain any zeros.
- The eigenvalues of the transfer function were real, nonpositive, and distinct.
- Control was effected by a single control variable $u(t)$, which was bounded in magnitude.
- The desired terminal state was the origin, which was an equilibrium point of the system. Upon reaching the origin the control needed to be shut off in order to maintain the system at the origin. [Ref. 1]

The method which we used to obtain the time-optimal control law was almost the same for each of these systems. Essential steps in our synthesis of the control were,

- We first reduced the system differential equation to a set of first order equations.
- We then chose a convenient set of state variables by means of a series of linear transformations which reduced the system matrix to its Jordan canonical form.
- We examined the Hamiltonian, and found the control which absolutely minimized the Hamiltonian. We observed that the time-optimal control had to be piecewise constant and could switch at most $n-1$ times for an nth order system.
- We then determined the control sequences which were candidates for time-optimal control.
- We used a method of elimination to determine the time-optimal control. We found a unique control sequence from among the candidates which would force a given state to the origin. Then we developed the control law. [Ref. 1]

The complexity of the controller may increase rapidly with the increase in the order of the system. For systems whose order is higher than three, some iterative procedure must be used to solve the system of transcendental equations that describe the switching hypersurface. Even though the equation of the switch hypersurface is complex, from a conceptual point of view the operation of a high-order time-optimal system presents no particular difficulty. Quite often, knowledge of the optimal solution can help the designer to construct an excellent suboptimal system. [Ref. 1]

Negative time approach was used in Ref. 5 to determine the time-optimal control for a third-order system with two integrators and a single time constant. The method requires analytic calculations of boundary conditions for each different set of eigenvalues. The process of elimination among the candidates for time-optimal control reduces the complexity and applicable to the higher-order systems as demonstrated in previous chapter.

APPENDIX

PROGRAM CODE

1. OPTIMALX.M

```
% This program simulates the 2nd order bang-bang controller  using a switching law  for
% the control effort
clear,clc,clg
% Setting eigenvalues of the system
alfa=1;  beta=2;
% Ratio of the Eigenvalues
k=(-beta)/(-alfa);
% State Equations for the uncoupled  system (x variables)
XA=[-alfa 0;0 -beta];   XB=[-alfa -beta]';
% State Equations for the uncoupled  system (z variables)
ZA=[-alfa 0;0 -beta];   ZB=[(-1/(alfa-beta))  (1/(alfa-beta))]';
% State Equations for the coupled  system (y variables)
YA=[0  1;-alfa*beta   -alfa-beta];   YB=[0 1]';
% Transition matrix between y (coupled) and z (uncoupled) systems
P=[1 1;-alfa -beta];
% Simulation time
tf=2.232;
% Time increment and number of steps for simulation
```

```
dt=0.001;   kmax=tf/dt+1;

% Equation of the switching curve for uncoupled system(x variables)

x1=-1.5:0.01:1.5;

x2=(x1./abs(x1)).*((1+abs(x1)).^k-1);

% Equation of the switching curve for uncoupled system(z variables)

c=alfa*(alfa-beta);

z1=x1;   z2=((c.*z1)./(-beta*(alfa-beta)*(abs(c.*z1))).*((1+abs(c.*z1)).^k-1));

% Equation of the switching curve for coupled system(y variables)

y1=(1/(alfa*(beta^2-alfa*beta))).*(alfa.*x2-beta.*x1);   y2=(1/(alfa-beta)).*(x2-x1);

x=zeros(2,kmax);   z=zeros(2,kmax);

y=zeros(2,kmax);   u=zeros(1,kmax);

time=zeros(1,kmax);

% Initial conditions

x(:,1)=[3 2]';

z(:,1)=[1/(alfa*(alfa-beta))*x(1,1) -1/(beta*(alfa-beta))*x(2,1)]';

y(:,1)=P*z(:,1);

% Discretize the Systems

[phi,del]=c2d(XA,XB,dt);   [phiz,delz]=c2d(ZA,ZB,dt);   [phiy,dely]=c2d(YA,YB,dt);

% Begin Simulation

for i= 1:kmax-1

u(i)=sign(x(1,i)/abs(x(1,i))*((1+abs(x(1,i)))^k-1)-x(2,i));

x(:,i+1)=phi*x(:,i)+del*u(i);
```

```
z(:,i+1)=phiz*z(:,i)+delz*u(i);

y(:,i+1)=phiy*y(:,i)+dely*u(i);

time(i+1)=time(i)+dt;

end

figure(1); plot(x1,x2,'r'); xlabel('X1');ylabel('X2');

title('SWITCHING CURVE FOR UNCOUPLED SYSTEM (X VARIABLES)');

figure(2);plot(y1,y2,'m');xlabel('Y1');ylabel('Y2');

title('SWITCHING CURVE FOR COUPLED SYSTEM (Y VARIABLES)');

figure(3); plot(time,u);grid; xlabel('TIME (Seconds');ylabel('MAGNITUDE');

title('CONTROL EFFORT vs TIME');

axis([0 max(time)+0.05 -1.75 1.75]);

figure(4); plot(x(1,:),x(2,:));grid ; xlabel('X1');ylabel('X2');

title('STATE TRAJECTORIES AND SWITCHING LINE (X VARIABLES)');

hold on; plot(x1,x2,'r:'); hold off

figure(5)

plot(y(1,:),y(2,:)); grid; xlabel('Y1');ylabel('Y2');

title('STATE TRAJECTORIES AND SWITCHING LINE FOR COUPLED SYSTEM

(Y VARIABLES)');

hold on ; plot(y1,y2,'r:');

hold off
```

2. RELAY3BT.M

```
% This program simulates the time optimal control of a third order system having three

%real   distinct negative  eigenvalues

% Written by Serhat Balkan                                        12 April 1994

clc,clear,clg,! del *.met,

% Set the eigenvalues of the system

alfa=1;  beta=2;  gama=3;

% Transition matrix between y (coupled) and z (uncoupled) systems

P=[(1/((beta-alfa)*(gama-alfa)))  (1/((alfa-beta)*(gama-beta))).......

 (1/((alfa-gama)*(beta-gama)));

  (alfa/((alfa-beta)*(gama-alfa))) (beta/((alfa-beta)*(beta-gama)))...

  (gama/((alfa-gama)*(gama-beta)));

  (alfa^2/((beta-alfa)*(gama-alfa))) (beta^2/((alfa-beta)*(gama-beta)))....

  (gama^2/((alfa-gama)*(beta-gama)))];

% State Equations for the uncoupled system (x variables)

XA=[-alfa  0  0;0 -beta 0;0  0 -gama];   XB=[alfa beta gama]';

% State Equations for the uncoupled system (z variables)

ZA=[-alfa 0 0;0 -beta 0;0 0 -gama];   ZB=[1  1  1]';

% State Equations for the coupled system (y variables)

YA=[ 0 1  0; 0  0 1;

   -alfa*beta*gama  -(alfa*beta+alfa*gama+gama*beta)  -(alfa+beta+gama)];

YB=[0 0 1]';
```

```
% Simulation time

tf=2.66;

% Time increment and number of steps for the simulation

dt=0.01;   kmax=tf/dt+1;   time=0;

% Discretize the systems

[phix,delx]=c2d(XA,XB,dt);   [phiz,delz]=c2d(ZA,ZB,dt);   [phiy,dely]=c2d(YA,YB,dt);

% Set initial conditions

x(:,1)=[2  3  4]';

z(:,1)=[(1/alfa)*x(1,1)  (1/beta)*x(2,1)  (1/gama)*x(3,1)]';

y(:,1)=P*z(:,1);

i=0;

% Order of the system

N=3;

% From Eqn.(4.35) set:

x1=x(1,1);   x2=x(2,1);

% Call function to determine w1, w2 and the optimal control

[w1,w2,deltas]=solve1b(x1,x2);

% Calculate  the third  point on the switching surface

x31=deltas*(-1+2*w1^gama-w2^gama);

% Decide whether initial state above or below the switching surface

m=x(3,1)-x31;

% Find the optimal control which drives the states to the switching
```

```
% surface in min. time when states are above the switching surface
while m > 0;
      i=i+1;
   u(i)=(-1^N);
      x(:,i+1)=phix*x(:,i)+delx*u(i);
   z(:,i+1)=phiz*z(:,i)+delz*u(i);
   y(:,i+1)=phiy*y(:,i)+dely*u(i);
       time(i+1)=time(i)+dt;
      x1=x(1,i+1);
      x2=x(2,i+1);
   [w1,w2,deltas]=solve1b(x1,x2);
      x31=deltas*(-1+2*w1^gama-w2^gama);
      m=x(3,i+1)-x31;
end;
% Find the optimal control which drives the states to the switching surface
% in min. time  when states are below the switching surface
while m < 0;
      i=i+1;
   u(i)=-(-1^N);
      x(:,i+1)=phix*x(:,i)+delx*u(i);
   z(:,i+1)=phiz*z(:,i)+delz*u(i);
   y(:,i+1)=phiy*y(:,i)+dely*u(i);
```

```
      time(i+1)=time(i)+dt;
      x1=x(1,i+1);
      x2=x(2,i+1);
   [w1,w2,deltas]=solve1b(x1,x2);
      x31=deltas*(-1+2*w1^gama-w2^gama);
      m=x(3,i+1)-x31;
end;
% Find the optimal control which drives the states to the switching
% curvce in min. time when states are on the switching surface
if m==0
      counter=1;
else
   counter=i;
end
for i=counter:kmax-1;
      x1=x(1,i);
   x2=x(2,i);
   [w1,w2,deltas]=solve1b(x1,x2);
 if (deltas*x(1,i)-1-2*w1+w2 == 0) &....
 (deltas*x(2,i)-1-2*w1^2+w2^2 ==0 &  deltas*x(3,i)-1-2*w1^gama+w2^gama == 0);
     u(i)=-(-1^N)*deltas;
          else
```

```
        u(i)=(-1^N)*deltas;
        end
        x(:,i+1)=phix*x(:,i)+delx*u(i);
    z(:,i+1)=phiz*z(:,i)+delz*u(i);
    y(:,i+1)=phiy*y(:,i)+dely*u(i);
        time(i+1)=time(i)+dt;
end
% Plot the outputs
plot3d(x(1,:),x(2,:),x(3,:),-45,45);
title('3-D PLOT OF THE STATE TRAJECTORIES FOR UNCOUPLED SYSTEM
(X VARIABLES)'),
meta 3ax; pause,clg,
plot(time,x(1,:),time,x(2,:),time,x(3,:));grid,
title('STATE TRAJECTORIES vs TIME (X VARIABLES)');
xlabel('TIME (seconds)');ylabel('MAGNITUDE');
meta 3bx,pause,clg ;axis([0 max(time)+0.1 -1.75 1.75]);
plot(time(1:length(u)),u);grid,
xlabel('TIME (seconds)');ylabel('MAGNITUDE');
title('OPTIMAL CONTROL EFFORT (u(t)) vs TIME'),
meta 3c
axis('normal'); pause,
plot3d(y(1,:),y(2,:),y(3,:),75,-45);
```

```
title('3-D PLOT OF THE STATE TRAJECTORIES FOR COUPLED SYSTEM
(Y VARIABLES)');
meta 3ay; pause,clg,
plot(time,y(1,:),time,y(2,:),time,y(3,:));grid,
title('STATE TRAJECTORIES vs TIME (Y VARIBLES)');
xlabel('TIME (seconds)');ylabel('MAGNITUDE');
meta 3by,pause,clg
```

3. SOLVE1B.M

```
% Function solve1b decides the optimal control and the required time to drive the states
% to the switching surface or to the  switching curve and passes these values to the
% main program (relay3bt.m).
% Written by Serhat Balkan                                              12 April 1994
function  [w1,w2,deltas] = solve1b(x1,x2)
delta=1;
% a1, a2, and delta are the local variables corresponding to w1, w2, and deltastar
%respectively.
[a1,a2,delta]=solve2b(x1,x2,delta);
% Check  if Eqn.(4.31) is satisfied or not
 if (a1>1) & (a2>a1)
    w1=a1;     w2=a2;
    deltas=delta;
 else
    delta=-1;
    [a1,a2,delta]=solve2b(x1,x2,delta);
    w1=a1;
    w2=a2;
    deltas=delta;
 end
end
```

4. SOLVE2B.M

```
% Function Solve2b calculates the required time to drive the states  to the switching
 % surface or to the switching curve processing  the given states and passes these values
% to the function solve1b
% Written by Serhat Balkan                                        12 April 1994
function  [a1,a2,delta] = solve2b(x1,x2,delta);
% Use the first two equations of Eqn.(4.33) to find w1, and w2 (a1 and a2 corresponds to
% w1 and w2 respectively)
 a1=delta*x1+1+0.5*sqrt(2*(x1^2)+4*delta*x1-2*delta*x2);
 a2=2*a1-delta*x1-1;
     % Eliminating any complex value
     if imag(a1) ~= 0;
       delta=-delta;
       a1=delta*x1+1+0.5*sqrt(2*(x1^2)+4*delta*x1-2*delta*x2);
       a2=2*a1-delta*x1-1;
     end
```

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